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# A ROBUST DISCRETIZATION OF THE REISSNER–MINDLIN PLATE WITH ARBITRARY POLYNOMIAL DEGREE*

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**Abstract**

A numerical scheme for the Reissner–Mindlin plate model is proposed. The method is based on a discrete Helmholtz decomposition and can be viewed as a generalization of the nonconforming finite element scheme of Arnold and Falk [SIAM J. Numer. Anal., 26(6):1276–1290, 1989]. The two unknowns in the discrete formulation are the in-plane rotations and the gradient of the vertical displacement. The decomposition of the discrete shear variable leads to equivalence with the usual Stokes system with penalty term plus two Poisson equations and the proposed method is equivalent to a stabilized discretization of the Stokes system that generalizes the Mini element. The method is proved to satisfy a best-approximation result which is robust with respect to the thickness parameter $t$.

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*Key words:* Reissner–Mindlin plate, Nonconforming finite element, Discrete Helmholtz decomposition, Robustness.

## 1. Introduction

The transverse displacement $w$ of a thin elastic plate of thickness $t > 0$ whose mid-surface is a bounded, open, simply connected, polygonal Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ and the rotation $\phi$ of the plate's fibers normal to the mid-surface are described by the Reissner–Mindlin plate model. Given a (re-scaled) transverse load $f \in L^2(\Omega)$, the Reissner–Mindlin plate problem with clamped boundary condition seeks $w \in H_0^1(\Omega)$ and $\phi \in \Phi := [H_0^1(\Omega)]^2$ such that

$$a(\phi, \psi) + \lambda t^{-2}(\nabla w - \phi, \nabla v - \psi)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for all } (v, \psi) \in H_0^1(\Omega) \times \Phi. \tag{1.1}$$

Here, the bilinear form $a(\cdot,\cdot)$ is defined by $a(\phi,\psi) := (\varepsilon(\phi), \mathbb{C}\varepsilon(\psi))_{L^2(\Omega)}$ for the linear green strain $\varepsilon(\cdot) = \operatorname{sym} D(\cdot)$ and the linear elasticity tensor $\mathbb{C}$ that acts on any symmetric matrix $A \in \mathbb{R}^{2\times 2}$ as follows

$$\mathbb{C}A = \frac{E}{12(1-\nu^2)}((1-\nu)A + \nu \operatorname{tr}(A) I_{2\times 2}).$$

For isotropic materials it is determined by Young's modulus $E > 0$ and the Poisson ratio $0 < \nu < 1/2$. Those also determine the constant $\lambda$ in (1.1), which reads $\lambda = (1+\nu)^{-1}E\kappa/2$ with a shear correction factor $\kappa$ usually chosen as $5/6$. More details on the mathematical model can be found in [5, 6].

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Standard schemes are known to exhibit *shear locking* and yield poor results for small thickness $t \ll h$. The reader is referred to [6,9] and the references therein for an overview of numerical schemes. It was the observation of Brezzi and Fortin [8] that the Helmholtz decomposition of the shear variable $\zeta := t^{-2}(\nabla w - \phi)$ may serve as the key for the robust numerical approximation of (1.1). Arnold and Falk [3] discovered a discrete analogue to that decomposition which led to a robust nonconforming finite element discretisation. Their discrete Helmholtz decomposition turned out useful for other purposes, too; but in its original form it is restricted to piecewise affine finite element functions and so to the lowest-order case. In [13] the generalization of the discrete Helmholtz decomposition to higher polynomial degrees from [14] was combined with the Taylor–Hood element [5] and optimal-order convergence rates were proved for the rotation variable through a superconvergence analysis. In this article we present and analyze the generalization of Arnold and Falk's scheme to higher polynomial degrees. This involves higher-order analogues of the Mini element. We formulate the new scheme in §2 and give robust a priori error estimates in §3. The numerical experiments of §4 investigate the performance of the method.

Standard notation on Lebesgue and Sobolev spaces applies throughout this paper. The $L^2$ inner product is denoted by $(v, w)_{L^2(\Omega)}$. The space of $L^2(\Omega)$ functions with vanishing global average reads $L^2_0(\Omega)$. For a function $v$ and a vector field $\psi$, the following differential operators are defined

$$\operatorname{div} \psi = \partial_1\psi_1 + \partial_2\psi_2, \quad \operatorname{rot} \psi = \partial_1\psi_2 - \partial_2\psi_1, \quad \operatorname{Curl} v = \begin{pmatrix} -\partial_2 v \\ \partial_1 v \end{pmatrix}.$$

The notation $A \lesssim B$ abbreviates $A \leq CB$ for some constant $C$ that is independent of the mesh size and the plate's thickness $t$.

## 2. The Method

This section is devoted to the precise definition of the novel method in Section 2.1. The discretization space for $\phi$ is stabilized with local bubble functions. Those can be condensated in the resulting system matrix. This is explained in more detail in Section 2.2.

### 2.1. Definition of the method

The new numerical scheme for Reissner–Mindlin plates is based on an equivalent reformulation of the original problem (1.1) based on the space of gradients $Z := \nabla H^1_0(\Omega)$. We assume that $\Omega$ is simply connected. With the spaces $X := [L^2(\Omega)]^2$ and $Q := H^1(\Omega) \cap L^2_0(\Omega)$, the Helmholtz decomposition gives the following characterization

$$Z = \{\sigma \in X \mid (\sigma, \operatorname{Curl} q)_{L^2(\Omega)} = 0 \text{ for all } q \in Q\}.$$

Note that, in two dimensions, the Curl operator is only the rotated gradient and therefore $H^1(\Omega)$ equals the space of all $L^2(\Omega)$ functions whose Curl is in $L^2(\Omega)$. Let $\eta \in H(\operatorname{div}, \Omega)$ be given with $-\operatorname{div} \eta = f$. The integration by parts and the substitutions $\sigma := \nabla w$ and $\tau := \nabla v$ show that (1.1) is equivalent to finding $\sigma \in Z$ and $\phi \in \Phi$ such that

$$a(\phi, \psi) + \lambda t^{-2}(\phi - \sigma, \psi - \tau)_{L^2(\Omega)} = (\eta, \tau)_{L^2(\Omega)} \quad \text{for all } (\tau, \psi) \in Z \times \Phi. \tag{2.1}$$

While on the continuous level this is merely a reformulation of (1.1), discretizations based on (2.1) turn out to benefit from an intrinsic discrete Helmholtz decomposition.