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AN EFFICIENT ADER DISCONTINUOUS GALERKIN SCHEME FOR DIRECTLY SOLVING HAMILTON-JACOBI EQUATION*

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Abstract

This paper proposes an efficient ADER (Arbitrary DERivatives in space and time) discontinuous Galerkin (DG) scheme to directly solve the Hamilton-Jacobi equation. Unlike multi-stage Runge-Kutta methods used in the Runge-Kutta DG (RKDG) schemes, the ADER scheme is one-stage in time discretization, which is desirable in many applications. The ADER scheme used here relies on a local continuous spacetime Galerkin predictor instead of the usual Cauchy-Kovalewski procedure to achieve high order accuracy both in space and time. In such predictor step, a local Cauchy problem in each cell is solved based on a weak formulation of the original equations in spacetime. The resulting spacetime representation of the numerical solution provides the temporal accuracy that matches the spatial accuracy of the underlying DG solution. The scheme is formulated in the modal space and the volume integral and the numerical fluxes at the cell interfaces can be explicitly written. The explicit formulae of the scheme at third order is provided on two-dimensional structured meshes. The computational complexity of the ADER-DG scheme is compared to that of the RKDG scheme. Numerical experiments are also provided to demonstrate the accuracy and efficiency of our scheme.

Mathematics subject classification: 65M06, 35F21, 70H20.
Key words: Hamilton-Jacobi equation, ADER, Discontinuous Galerkin methods, Local continuous spacetime Galerkin predictor, High order accuracy.

1. Introduction

Consider the Hamilton-Jacobi (HJ) equation

$$\varphi_t + H(\nabla_{\boldsymbol{x}}\varphi, \boldsymbol{x}) = 0, \quad \varphi(\boldsymbol{x}, 0) = \varphi^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \in \mathbb{R}^d, \tag{1.1}$$

with suitable boundary conditions, where $H(\cdot)$ denotes the Hamiltonian. The HJ equations are used in many application areas, such as optimal control theory, geometrical optics, crystal growth, image processing and computer vision. The solutions of such equations are continuous but their derivatives could be discontinuous even if the initial condition is smooth. Viscosity solutions were firstly introduced and studied in [6, 7], which are the unique physically relevant solutions.

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It is well known that the HJ equations are closely related to hyperbolic conservation laws, thus many successful numerical methods for the conservation laws can be adapted for solving the HJ equations. In [7], a monotone finite difference scheme was introduced and proved to be convergent to the viscosity solution. A second order finite difference essentially non-oscillatory (ENO) scheme was developed in [16], and then a higher-order weighted ENO (WENO) scheme is proposed in [14]. Tang and his collaborators developed an adaptive mesh redistribution method and the relaxing scheme for the HJ equations [18, 19]. Qiu et al. [17] developed the Hermite WENO (HWENO) schemes of the HJ equations. The high order finite difference WENO scheme on unstructured meshes was developed in [22], but its implementation is a bit complicated.

Alternatively, a DG method was designed in [13] to solve the HJ equations, and its reinterpretation and simplified implementation was given in [15]. Those DG methods were based on the fact that the derivatives of the solution satisfied the conservation laws. It was correct in the one-dimensional case but at risk in the multi-dimensional case because corresponding multi-dimensional conservation laws is only weakly hyperbolic in general. Later, a DG method for directly solving the HJ equations with convex Hamiltonians was proposed in [3]. It was further improved and a new DG method was derived for directly solving the general HJ equations with nonconvex Hamiltonians in [4]. This paper will construct the scheme based on the RKDG scheme in [4]. The RKDG method [5] was originally designed to solve conservation laws, which has the advantages of flexibility on complicated geometries and a compact stencil, and is easy to obtain high order accuracy.

Most of the above methods use the multi-stage Runge-Kutta time discretization, thus have the advantage of simplicity but are time-consuming because at each stage, the volume integration and the numerical fluxes at cell interfaces have to be calculated and the nonlinear limiters should be performed to suppress the numerical oscillations. Thus, in order to save the computational cost, it is desirable to use an alternative to the multi-stage Runge-Kutta method. One choice is the Lax-Wendroff type time discretization, which converts all (or partial, when approximations with certain accuracy are expected) time derivatives in a temporal Taylor expansion of the solution into spatial derivatives by repeatedly using the underlying differential equation and its differentiated forms [12]. In [12], a local-structure-preserving DG method with Lax-Wendroff type time discretization was proposed for solving the HJ equations. It is shown that such method is relatively more efficient than the RKDG method in [15]. But the Cauchy-Kowalewski procedure may become a little cumbersome when we want to construct a high order scheme. This paper will use the time discretization (named ADER) proposed in [8, 9]. The ADER scheme has been successfully applied to the (magneto) hydrodynamics and relativistic (magneto) hydrodynamics with stiff or non-stiff source terms [1,2,8,9,11]. It is based on a local spacetime Galerkin predictor step, at which a local Cauchy problem is solved in each cell, based on a weak formulation of the original partial differential equations in spacetime. Through the above procedure, the resulting spacetime representation of the numerical solution provides the temporal accuracy that matches the spatial accuracy of the underlying DG solution. The ADER scheme is a one-step one-stage time discretization, which means that the volume integration and the numerical fluxes terms at cell interfaces are only calculated once at each time step. Our ADER-DG scheme is formulated in modal space. Thanks to the spacetime representation of the numerical solution, we can write down explicit formulae of the scheme using the strategy presented in [1], and we will provide the implementation details of the scheme at third order on two-dimensional structured meshes. Our ADER-DG scheme can capture the viscosity solution accurately and efficiently, and will be validated by the analysis of the computational complexity