The Value Distribution and Normality Criteria of a Class of Meromorphic Functions

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Abstract: In this article, we use Zalcman Lemma to investigate the normal family of meromorphic functions concerning shared values, which improves some earlier related results.

Key words: meromorphic function, shared value, normal criterion

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1 Introduction and Main Results

Let $D$ be a domain of the open complex plane $\mathbb{C}$, $f(z)$ and $g(z)$ be two nonconstant meromorphic functions defined in $D$, $a$ be a finite complex value. We say that $f$ and $g$ share $a$ CM (or IM) in $D$ provided that $f - a$ and $g - a$ have the same zeros counting (or ignoring) multiplicity in $D$. When $a = \infty$, the zeros of $f - a$ means the poles of $f$ (see [1]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory (see [2]–[4]).

It is also interesting to find normality criteria from the point of view of shared values. In this area, Schwick[5] first proved an interesting result that a family of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivative. And later, more results about shared values' normality criteria related a Hayma conjecture of higher derivative have emerged (see [6]–[13]).

Lately, Chen[14] proved the following theorems.

Theorem 1.1 Let $D$ be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in $D$. Let $k, n, d \in \mathbb{N}_+$, $n \geq 3$, $d \geq \frac{k + 1}{n - 2}$ and $a, b$ be two finite complex numbers with...
Suppose that every \( f \in \mathcal{F} \) has all its zeros of multiplicity at least \( k \) and all its poles of multiplicity at least \( d \). If \( f^{(k)} - af^n \) and \( g^{(k)} - ag^n \) share the value \( b \) IM for every pair of functions \((f, g)\) of \( \mathcal{F} \), then \( \mathcal{F} \) is a normal family in \( D \).

**Theorem 1.2** \( \quad \) Let \( D \) be a domain in \( \mathbf{C} \) and let \( \mathcal{F} \) be a family of meromorphic functions in \( D \). Let \( k \in \mathbf{N}_+ \) and \( a, b \) be two finite complex numbers with \( a \neq 0 \). Suppose that every \( f \in \mathcal{F} \) has all its zeros of multiplicity at least \( k + 1 \) and all its poles of multiplicity at least \( k + 2 \). If \( f^{(k)} - af^2 \) and \( g^{(k)} - ag^2 \) share the value \( b \) IM for every pair of functions \((f, g)\) of \( \mathcal{F} \), then \( \mathcal{F} \) is a normal family in \( D \).

A natural problem arises: what can we say if \( f^{(k)} - af^n \) in Theorem 1.1 is replaced by the \((f^{(k)})^m - af^n\)? In this paper, we prove the following results.

**Theorem 1.3** \( \quad \) Let \( D \) be a domain in \( \mathbf{C} \) and let \( \mathcal{F} \) be a family of meromorphic functions in \( D \). Let \( k, n, m, d \in \mathbf{N}_+ \), \( n \geq m + 2, d \geq \frac{mk + 1}{n - m - 1} \) and \( a, b \) be two finite complex numbers with \( a \neq 0 \). Suppose that every \( f \in \mathcal{F} \) has all its zeros of multiplicity at least \( k + 1 \) and all its poles of multiplicity at least \( mk + 2 \). If \((f^{(k)})^m - af^{m+1}\) and \((g^{(k)})^m - ag^{m+1}\) share the value \( b \) IM for every pair of functions \((f, g)\) of \( \mathcal{F} \), then \( \mathcal{F} \) is a normal family in \( D \).

**Theorem 1.4** \( \quad \) Let \( D \) be a domain in \( \mathbf{C} \) and let \( \mathcal{F} \) be a family of meromorphic functions in \( D \). Let \( k, m \in \mathbf{N}_+ \) and \( a, b \) be two finite complex numbers with \( a \neq 0 \). Suppose that every \( f \in \mathcal{F} \) has all its zeros of multiplicity at least \( k + 1 \) and all its poles of multiplicity at least \( mk + 2 \). If \((f^{(k)})^m - af^{m+1}\) and \((g^{(k)})^m - ag^{m+1}\) share the value \( b \) IM for every pair of functions \((f, g)\) of \( \mathcal{F} \), then \( \mathcal{F} \) is a normal family in \( D \).

## 2 Some Lemmas

**Lemma 2.1** \( ^{[15]} \) \( \quad \) Let \( \mathcal{F} \) be a family of meromorphic functions on the unit disc satisfying all zeros of functions in \( \mathcal{F} \) have multiplicity \( \geq p \) and all poles of functions in \( \mathcal{F} \) have multiplicity \( \geq q \). Let \( \alpha \) be a real number satisfying \(-q < \alpha < p \). Then \( \mathcal{F} \) is not normal at 0 if and only if there exist

- a) a number \( 0 < r < 1 \);
- b) points \( z_n \) with \( |z_n| < r \);
- c) functions \( f_n \in \mathcal{F} \);
- d) positive numbers \( \rho_n \to 0 \)

such that \( g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \) converges spherically uniformly on each compact subset of \( \mathbf{C} \) to a non-constant meromorphic function \( g(\zeta) \), whose all zeros have multiplicity \( \geq p \) and all poles have multiplicity \( \geq q \) and order is at most 2.

**Lemma 2.2** \( \quad \) Let \( f(z) \) be a meromorphic function such that \( f^{(k)}(z) \neq 0 \) and \( a \in \mathbf{C} \setminus \{0\} \), \( k, m, n, d \in \mathbf{N}_+ \) with \( n \geq m + 2, d \geq \frac{km + 1}{n - m - 1} \). If all zeros of \( f \) are of multiplicity at least