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On the Reducibility of a Class of Linear Almost Periodic Differential Equations

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Abstract: In this paper, we use KAM methods to prove that there are positive measure Cantor sets such that for small perturbation parameters in these Cantor sets a class of almost periodic linear differential equations are reducible.

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1 Introduction and the Main Result

This paper considers the reducibility of the following system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = [\boldsymbol{A} + \varepsilon \boldsymbol{Q}(t)]\,\boldsymbol{x}, \tag{1.1}$$

where $\boldsymbol{A}$ is an $r \times r$ constant matrix, $\boldsymbol{Q}(t)$ is an $r \times r$ almost periodic matrix with respect to t, and ε is a small perturbation parameter.

We say that a function f is a quasiperiodic function of time t with basic frequencies $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_d)$, if $f(t) = F(\theta_1, \theta_2, \cdots, \theta_d)$, where F is 2π periodic in all its arguments and $\theta_n = \omega_n t$ for $n = 1, 2, \cdots, d$. f is called analytic quasiperiodic in a strip of width ρ if F is analytical on

$$D_\rho = \{\boldsymbol{\theta} \mid |\Im\theta_m| \leq \rho,\ m = 1, 2, \cdots, r\}.$$

In this case we denote the norm by

$$\|f\|_\rho = \sum_{k\in\mathbf{Z}^d} |F_k| \mathrm{e}^{\rho|k|}.$$

A function f is almost periodic, if $f(t) = \sum\limits_{n=1}^{\infty} f_n(t)$, where $f_n(t)$ are all quasiperiodic for $n = 1, 2, \cdots$.

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A change of variables $\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{y}$ is a Lyapunov-Perron (LP) transform if $\boldsymbol{P}$ is non-singular, and $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ and $\dot{\boldsymbol{P}}$ are bounded. Moreover, if $\boldsymbol{P}$, $\boldsymbol{P}^{-1}$ and $\dot{\boldsymbol{P}}$ are almost periodic, the change $\boldsymbol{x} = \boldsymbol{P}(t)\boldsymbol{y}$ is called almost periodic LP transformation. If there is an almost periodic LP transformation changing the equation (1.1) into $\dot{\boldsymbol{y}} = \boldsymbol{B}\boldsymbol{y}$, the equation (1.1) is called reducible.

If $\boldsymbol{Q} = (q_{mn})$ is periodic the reducibility in all cases is given by the classical Floquet theory. If $\boldsymbol{Q} = (q_{mn})$ is quasiperiodic and the eigenvalues of $\boldsymbol{A}$ are all different, Jorba-Simó[1] proved that if the eigenvalues of $\boldsymbol{A}$ and the frequencies of $\boldsymbol{Q} = (q_{mn})$ satisfy some non-resonant conditions and non-degeneracy conditions, there is a positive measure Cantor set E such that for $\varepsilon \in E$ the equation (1.1) is reducible. Xu[2] proved the similar result when $\boldsymbol{Q} = (q_{mn})$ is quasiperiodic and the eigenvalues of $\boldsymbol{A}$ are multiple. If $\boldsymbol{Q} = (q_{mn})$ is almost periodic, the reducible problem seems difficult to study. The difficulty comes from the description of related "non-resonant condition" for the infinitely many frequencies. Xu and You[3], under the " spacial structure" described in [4] and some non-resonant conditions, obtained reducible result for (1.1) by KAM method when the eigenvalues of $\boldsymbol{A}$ are all different. In this paper, we are going to study the reducibility for the system (1.1) when $\boldsymbol{Q} = (q_{mn})$ is almost periodic and the eigenvalues of $\boldsymbol{A}$ are multiple.

Now let us introduce the "space structure" and "approximation function" and some related definitions.

Definition 1.1[4] *Let τ consist of the subsets of natural numbers set $\mathbf{N}$. $(\tau, [\,\cdot\,])$ is called finite spacial structure in $\mathbf{N}$, if τ satisfies*

(1) $\emptyset \in \tau$;

(2) *if* $\Lambda_1, \Lambda_2 \in \tau$, *then* $[\Lambda_1 \cup \Lambda_2] \le [\tau]$;

(3) $\bigcup\limits_{\Lambda\in\tau} \Lambda = \mathbf{N}$.

And $[\,\cdot\,]$ is a weight function, i.e., $[\emptyset] = 0$, $[\Lambda_1 \cup \Lambda_2] \le [\Lambda_1] + [\Lambda_2]$.

Definition 1.2 *Let $\boldsymbol{k} \in \mathbf{Z}^{\mathbf{N}}$. Denote the support set of $\boldsymbol{k}$ by*

$$\mathrm{supp}\boldsymbol{k} = \{(n_1, n_2, \cdots, n_l) \mid k_m \ne 0,\ m = n_1, n_2, \cdots, n_l,\ k_m = 0,\ m = \textit{other number}\}.$$

Denote the weight value by

$$[\boldsymbol{k}] = \inf_{\mathrm{supp}\boldsymbol{k} \subset \Lambda, \Lambda \in \tau} [\Lambda].$$

Write $|\boldsymbol{k}| = \sum\limits_{i=1}^{\infty} |k_i|$.

Assume that $\boldsymbol{Q}(t) = (q_{mn}(t))$ is a quasiperiodic $r \times r$ matrix. If for all $m, n = 1, 2, \cdots, r$, $q_{mn}(t)$ are analytic on

$$D_\rho = \{\boldsymbol{\theta} \mid |\Im\theta_m| \le \rho,\ m = 1, 2, \cdots, r\},$$

then $\boldsymbol{Q}(t)$ is called analytic on the strip D_ρ. Denote the norm by

$$\|\boldsymbol{Q}(t)\|_\rho = r \times \max_{1\le m,n\le r} \|q_{mn}(t)\|_\rho.$$

If $\boldsymbol{Q}(t) = \sum\limits_{\Lambda\in\tau} \boldsymbol{Q}_\Lambda(t)$, where $\boldsymbol{Q}_\Lambda(t)$ are quasiperiodic matrices with basic frequencies $\boldsymbol{\omega}_\Lambda = \{\omega_i \mid i \in \Lambda\}$, then $\boldsymbol{Q}(t)$ is called almost periodic matrix with spatial structure $(\tau, [\,\cdot\,])$ and