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A Family of Nonconforming Rectangular Elements for Strain Gradient Elasticity

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Abstract. We propose a family of nonconforming rectangular elements for the linear strain gradient elastic model. Optimal error estimates uniformly with respect to the small material parameter have been proved. Numerical results confirm the theoretical prediction.

AMS subject classifications: 65N30, 65N15, 74K20

Key words: Nonconforming finite elements, strain gradient elasticity, uniform error estimate.

1 Introduction

Strain gradient models play an important role in the characterization of the heterogeneity and the size effect of materials down to micro scale. Though it may date back to Cosserat brothers' classical work [12], numerical simulation of this model is rather recent [14, 24, 27, 30] because strain gradient models are usually quite complicate. In particular, they contain a couple of materials parameters and standard finite element approximations usually do not work for such model. Aifantis et al. [2,26] proposed a linear strain gradient elastic model that has only one material parameter. This simplified model successfully eliminated the strain singularity of the brittle crack tip field [13].

From a mathematical point of view, Aifantis' model is a singularly perturbed elliptic system of fourth order due to the appearance of the strain gradient. A natural choice for such type problems is C^1 finite elements such as Argyris triangle [3], while the large number of local degrees of freedom and the high degree of polynomials used in the shape

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functions are the main impediments for practical simulation. In [21], the authors proposed two nonconforming elements with 21 degrees of freedom and even simpler nonconforming elements have been recently constructed in [22]. The simplest one among them is the tensor product of the Morley's triangle [23] with a modification of the elastic strain energy. All elements converge uniformly with respect to the small material parameter.

In contrast to the triangle elements in [21, 22], we consider the rectangular elements in the present work. The rectangular elements are equally powerful when they are combined with the isoparametric concept, which has been developed for Hermite elements in [25]. The famous Bogner-Fox-Schmit element (BFS) [6] has been exploited in [24,30] to approximate the strain gradient elastic model in two and three dimensions. The authors in [14] found that BFS outperforms several other elements in solving a nonlinear strain gradient elastic model. One drawback of BFS element is the large number of local degrees of freedom, the other is that the second order derivative appears in the definition of the degrees of freedom, which unfortunately brings in extra difficulty in dealing with boundary condition, because only the normal traction appears in the boundary condition of the strain gradient elastic model. To avoid such difficulty, we propose a family of rectangular element that is H^2 nonconforming while H^1 conforming. This means the finite element function is continuous, while the derivative is discontinuous across the element boundaries. Based on the discrete H^2 inequality proved recently in [22], the tensor product of this element may be used to approximate the strain gradient elastic model. We prove that this element converges with optimal rate for both the smooth solution and the solution with a strong boundary layer. The latter is very common in the strain gradient elastic model.

The proposed element consists of the serendipity family of finite element [11] augmented with a special bubble space. This bubble space is a natural extension of the one appeared in [19]. Using the orthogonal properties of the Jacobi polynomials [28], we derive the explicit basis function associated with the bubble space, while the explicit representation of the corresponding bubble space in [19] is still unknown. Such basis functions of the bubble space could be exploited to construct the basis functions of the proposed element. It is worth pointing out that such family of rectangular element naturally yields a hierarchical nonconforming plate bending element, which obviously makes the nonconforming elements more competitive compared with the conforming element [9,20,31] and the discontinuous Galerkin method; cf. [8]. It is worthwhile to mention that other orthogonal polynomials such as Legendre polynomials have been exploited to construct nonconforming element for plate bending element [17].

The structure of the paper is as follows. In Section 2, we introduce Aifantis' strain gradient model and the variational formulation. The finite element space is introduced in Section 3. Besides the structure of the element is clarified and the optimal approximation properties are proved for both smooth and nonsmooth functions. Error estimates are also proved in this part. The numerical results are reported in the last section. The explicit basis functions for the lowest order element and next to the lowest order element are