Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 6, pp. 1358-1375

DOI: 10.4208/aamm.OA-2018-0186
December 2019

Analysis of Finite Difference Approximations of an Optimal Control Problem in Economics

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Received 26 August 2018; Accepted (in revised version) 21 April 2019

Abstract. We consider an optimal control problem which serves as a mathematical model for several problems in economics and management. The problem is the minimization of a continuous constrained functional governed by a linear parabolic diffusion-advection equation controlled in a coefficient in advection part. The additional constraint is non-negativity of a solution of state equation. We construct and analyze several mesh schemes approximating the formulated problem using finite difference methods in space and in time. All these approximations keep the positivity of the solutions to mesh state problem, either unconditionally or under some additional constraints to mesh steps. This allows us to remove corresponding constraint from the formulation of the discrete problem to simplify its implementation. Based on theoretical estimates and numerical results, we draw conclusions about the quality of the proposed mesh schemes.

AMS subject classifications: 65M06, 65M12, 65M60

Key words: Mean field game, optimal control problem, parabolic diffusion-advection equation, finite difference methods.

1 Introduction

The theory of optimal control is involved in many models in economics. An actual and common method of solving such problems is mean field game (MFG) formulation. This approach was proposed in [1,2] and describes situations of equilibrium by considering a

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continuum of players (also called agents) through two forward/backward coupled PDE system. In this system the first equation is a transport equation (or, Fokker-Planck equation) describing the evolution of the distribution of agents, while the second equation corresponds to a Hamilton-Jacobi-Bellman equation derived from the optimization of a criterion by means of a control. MFG formulation of the problems are thoroughly used both for their theoretical study and numerical solution (see [3–8] and the bibliography therein).

The link between MGF and optimal control takes place in the so-called potential case (see [7] for the details). In this case equilibrium system solution of MFG is a critical point of an optimal control problem governed by transport equation.

In this paper we consider a problem of minimization of a cost functional

$$J(m,\alpha)=\int_0^T\int_0^1 \mathrm{e}^{-rt}m\big(f(m)+\frac{\alpha^2}{2}\big)dxdt, \qquad r=\text{const}\geq 0,$$

with respect to pair state-control (m,α), which satisfy a linear diffusion-advection state equation

$$\frac{\partial m}{\partial t}-\frac{\sigma^2}{2}\frac{\partial^2 m}{\partial x^2}-\frac{\partial}{\partial x}(\alpha m)=0$$

controlled in a coefficient of advection term. Some "economical meanings" of the problem and concrete examples of the functions in this formulation can be found in [8,9].

In the case of differentiable function $f(m)$ we can use Lagrange function to write first order optimality conditions for the considered minimization problem. It is just MFG formulation of the problem, which resulting system consists of two coupled forward-backward parabolic equations and a term that represents the derivative J'_α. In [8] this system was approximated by using finite elements in space and implicit scheme in time, and solved by an iterative solution method. In [9] the initial minimization problem was approximated and solved by a new monotone iterative algorithm. State equation–transport equation–was approximated by a finite difference scheme in space and by explicit approximation in time.

One of the most important property of the solutions m of any mesh approximation of state equation is its positivity. This corresponds to its meaning as the density of agents and, moreover, it is necessary from mathematical point of view, because the cost functional is not bounded from below for functions m which can have the negative values. In [9] this property is saved for the constructed approximation by introducing an additional constraint which connects mesh parameters and sought solution. In the article [8] there were no theoretical studies of the constructed approximations.

The aim of this article is construction and investigation theoretically and numerically several finite difference approximations of the mentioned above optimal control problem. To approximate state equation we use so-called summation equality in space variable and one of the following approximations in time: fully implicit (backward Euler), semi-implicit or fully explicit (forward Euler). The solutions of all constructed approximations