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A Weak Galerkin Method with C^0 Element for Fourth Order Linear Parabolic Equation

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Abstract. This paper is concerned with the C^0 weak Galerkin finite element method for a fourth order linear parabolic equation. The method is based on the construction of a discrete weak Laplacian operator. The error estimates are obtained for semi-discrete weak Galerkin finite element method. Numerical results are presented to confirm the theory.

AMS subject classifications: 60H15, 65C30, 65M60

Key words: Fourth order parabolic equation, weak Galerkin, error estimate.

1 Introduction

In this paper, we consider an initial-boundary value problem of fourth order linear parabolic equation

$$u_t+\Delta^2 u=f, \quad x\in\Omega, \quad 0\le t\le \bar{t}, \tag{1.1a}$$
$$u=\Delta u=0, \quad x\in\partial\Omega, \quad 0\le t\le \bar{t}, \tag{1.1b}$$
$$u(0,\cdot)=\psi, \quad x\in\Omega, \tag{1.1c}$$

where Δ is the Laplacian operator and Ω is an open, bounded set in $\mathbb{R}^d$ $(d=2,3)$ with Lipschitz-continuous boundary $\partial\Omega$. ψ is a given function on Ω.

Let $H=L^2(\Omega)$ with standard inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We also denote by $H^m=H^m(\Omega)$ the standard Sobolev space. The variational formulation for (1.1a)-(1.1c) is to find $u(t,x)\in C^1[0,\bar{t};H_0^2(\Omega))]$ and $u(0,\cdot)=\psi$ such that

$$(u_t,v)+(\Delta u,\Delta v)=\langle f,v\rangle, \quad \forall v\in H_0^2(\Omega). \tag{1.2}$$

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where $H_0^2(\Omega)$ is a subspace of the Sobolev space $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between H^{-2} and H_0^2.

Based on the variational form (1.2), one may design various conforming finite element schemes for (1.1a)-(1.1c) by constructing finite element spaces as subspaces of $H^2(\Omega)$. It is known that H^2-conforming method essentially requires C^1-continuous piecewise polynomials on a prescribed finite element partition. This would lead to a high polynomial degrees (see [2, 3]). For example, Argyris element need 21 degrees of freedom on each triangular element for $d=2$. Due to the complexity of construction of C^1 element, many nonconforming or discontinuous finite element methods have been developed for solving the biharmonic equation in [4–9, 21, 22]. C^0 interior penalty methods are studied in [1, 10], which are similar to our C^0-weak Galerkin method except there is no penalty parameter here.

In this paper, we introduce a weak Galerkin method with C^0 element for parabolic biharmonic equation with low solution regularities. Weak Galerkin finite element method, first introduced in [12] (see also [17] for extensions), uses nonconforming elements to relax the difficulty in the construction of conforming elements. Unlike the classical nonconforming finite element method where standard derivatives are taken on each element, the weak Galerkin finite element method relies on weak derives taken as approximate distributions for the functions in nonconforming finite element spaces. In general, weak Galerkin method refers to finite element techniques for partial differential equations in which differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms as distributions [15, 16, 18–20, 23]. A weak Galerkin method for the biharmonic equation has been derived in [12] by using totally discontinuous functions of piecewise polynomials on general partitions of arbitrary shape of polygons/polyhedra. The weak Galerkin method uses the discontinuous functions in the finite element procedure which endows the method with high flexibility to deal with geometric complexities and boundary conditions. Such a flexibility gives robustness in the enforcement of interface jump conditions for interface problem [11].

Our work is motivated by the recent paper [14]. In this paper, we will use the C^0 weak Galerkin method for fourth order linear parabolic equation (1.1a)-(1.1c) by redefining a weak Laplacian, denoted by Δ_w. Comparing with the weak Galerkin method developed in [13] and standard Galerkin finite element method, the C^0 weak Galerkin finite element formulation has less number of unknowns due to the continuity requirement. For example, there is only 12 degrees of freedom on each triangular element in 2D. On the other hand, due to the same continuity requirement, the C^0 weak Galerkin method allows only traditional finite element partitions (such as triangles/quadrilaterals in 2D), instead of arbitrary polygonal/polyhedral grids as allowed in [13]. The key of the method lies in the use of a discrete weak Laplacian plus a stabilization that is parameter-free. A suitably-designed interpolation operator is needed for the convergence analysis of the C^0 weak Galerkin formulation. Consequently, we show optimal order of convergence in L^2 norm.

The rest of the paper is organized as follows. In Section 2, we introduce the weak