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# A Novel Proof on the Existence of the Solution of Fractional Control Problem Governed by Burgers Equations

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**Abstract.** In this study, first we give some conditions to prove that a fractional Burgers equation has a unique solution. For this aim, we define a special operator to deduce the uniqueness of the solution. Then we prove that the optimal control problem under fractional Burgers equation has at least one optimal solution.

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**Key Words**: Burgers equation; fractional optimal control.

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## 1 Introduction

The analysis and applications of fractional partial differential equations [1] have enormously grown in many different fields. By using various tools from fractional calculus, we can obtain the models of many phenomena including fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences, more precisely and accurately [2, 3].

The fractional Burgers equation [4] illustrates the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in

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bubbly liquids. Biler et al. [5] studied local and global in time solutions to a class of multidimensional generalized Burgers-type equations with a fractal power of the Laplacian in the principal part and with general algebraic nonlinearity.

So far, many researchers have solved the fractional Burgers equations by various techniques such as homotopy analysis technique, operational matrix based method, and tau method [1]. Alam Khan et al. [6] utilized the homotopy perturbation method (HPM) and generalized differential transform method (GDTM) for solving time-fractional Burgers and coupled Burgers equations. Saad and Al-Sharif [7] studied approximate solutions using the variational iteration method (VIM) for the time and space-time fractional Burgers equation. Mao and Karniadakis [8] considered a new fractional viscous Burgers equation with nonlinear fractional terms which reduces to the standard Burgers equation for a suitable choice of the fractional orders. Kurt et al. [9] found the exact solution of a time fractional Burgers equation is in the sense of conformable fractional derivative, with dirichlet and initial conditions by Hopf-Cole transform. Al-Khaled [10] constructed the solution by using different approach that is based on using collocation techniques. The solution is based on using the Sinc method, which builds an approximate solution valid on the entire spatial domain, and in the time Domain. Author utilized the shifted Legendre polynomials to replace the time fractional derivatives.

In this paper, first we prove that a class of fractional Burgers equations under some special conditions has a unique solution. Then we investigate an optimal control problem associated to the considered class of fractional Burgers equations and show that this problem has at least one optimal solution.

The structure of this paper is as follows. In Section 2, we give some preliminaries which we need them to prove the main results in the paper. In Section 3, we introduce an optimal control problem for a class of fractional Burgers equations and give two important results related to this problem. Finally, conclusion is given in Section 4.

## 2 Preliminaries

**Definition 2.1.** ([11]) *Let $\mathbf{X}$ be a normed linear space. A linear functional $\mathbb{T}$ on $\mathbf{X}$ is said to be bounded if there is an $M \geq 0$ such that*

$$|\mathbb{T}(f)| \leq M\|f\| \quad \text{for any } f \in \mathbf{X}.$$

*The infimum of all such $M$ is called the norm of $\mathbb{T}$ and it is denoted by $\|\mathbb{T}\|_\star$. The collection of bounded linear functionals on $\mathbf{X}$ is denoted by $\mathbf{X}^\star$ and is called the dual space of $\mathbf{X}$ which is a linear space.*

**Definition 2.2.** ([11]) *The linear operator $\mathbb{J}: \mathbf{X} \to (\mathbf{X}^\star)^\star$ defined by*

$$\mathbb{J}(x)[\psi] = \psi(x) \quad for \quad all \quad x \in \mathbf{X}, \psi \in \mathbf{X}^\star,$$

*is called the natural embedding of $\mathbf{X}$ into $(\mathbf{X}^\star)^\star$. Also, the space $\mathbf{X}$ is said to be reflexive when $\mathbb{J}(\mathbf{X}) = (\mathbf{X}^\star)^\star$. It is customary to denote $(\mathbf{X}^\star)^\star$ by $\mathbf{X}^{\star\star}$ and call $\mathbf{X}^{\star\star}$ the bidual of $\mathbf{X}$.*