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ON THE VALIDITY OF THE LOCAL FOURIER ANALYSIS*

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Abstract

Local Fourier analysis (LFA) is a useful tool in predicting the convergence factors of geometric multigrid methods (GMG). As is well known, on rectangular domains with periodic boundary conditions this analysis gives the *exact* convergence factors of such methods. When other boundary conditions are considered, however, this analysis was judged as been heuristic, with limited capabilities in predicting multigrid convergence rates. In this work, using the Fourier method, we extend these results by proving that such analysis yields the exact convergence factors for a wider class of problems, some of which cannot be handled by the traditional rigorous Fourier analysis.

Mathematics subject classification: 65N55, 65T50
Key words: Local Fourier analysis, Multigrid, Fourier method.

1. Introduction

There exist two main approaches to quantitatively analyze the convergence of geometric multigrid algorithms: the rigorous Fourier analysis (RFA) [1, 2] (also called model problem analysis) and the local Fourier analysis [3] (or local mode analysis). Both techniques use error expansion in terms of the eigenvectors of a discrete differential operator, followed by study of the behavior of the multigrid error transfer operator when acting on these components. The main difference is that the RFA takes into account the boundary conditions, while the LFA neglects the effect of boundary conditions by assuming that the discrete differential operator is defined on an infinite grid. Clearly, in order to perform a RFA it is necessary to find a basis of eigenvectors for the discretized boundary value problem, that is, the basis elements must satisfy the boundary conditions. The convergence rates predicted by RFA then are exact, but also such procedure limits its applicability since to find such a basis may be impossible. Therefore, RFA gives the exact convergence rates of the GMG, but only for a small class of model problems. The LFA, on the other hand, works on an infinite grid and uses a basis of complex valued, exponential functions, which makes it applicable to a much wider class of discretized differential operators. It is well known [4] that the LFA provides accurate approximations of the

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asymptotic convergence factors of the GMG algorithms for many problems, and, moreover, it is exact for problems with periodic boundary conditions. In this work we focus on the question whether the LFA can be made exact (rigorous) for a wider class of discretized boundary value problems, not necessarily with periodic boundary conditions. As it turns out, we can answer this question positively. Our approach relies on the embedding of the model problem into a periodic problem. Similar ideas have also been explored in works on circulant preconditioners for elliptic problems [5, 6] and also for preconditioning the indefinite Helmholtz equation [7]. We introduce a class of operators called LFA-compatible operators here and prove that for such operators the LFA gives the exact multigrid convergence factors. Our studies include the Dirichlet, the Neumann and the mixed boundary condition problem for a constant coefficient, reaction-diffusion equation on a d-dimensional tensor product grid.

2. Preliminaries

2.1. The Dirichlet problem and its discretization

We consider a reaction-diffusion problem in d spatial dimensions on the domain $\Omega^D = (0,1)^d$,

$$-\Delta u(\boldsymbol{x}) + cu(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^D, \quad \text{and} \quad u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega^D, \tag{2.1}$$

where $c > 0$ is a constant. First, let us consider the simplest case when $d = 1$ (one dimensional problem). The computational domain then is the interval $\Omega^D = (0,1)$ and the corresponding two-point boundary value problem (2.1) is:

$$-u''(x) + cu(x) = f(x), \quad x \in \Omega^D, \quad u(0) = u(1) = 0. \tag{2.2}$$

For $d = 1$, we introduce a uniform grid $\Omega_h^D = \{x_k = kh\}_{k=0}^n$, with step size $h = 1/n$, $n \in \mathbb{N}$ and we discretize this problem by the standard central difference scheme. As a result, we obtain the linear system of algebraic equations with tri-diagonal matrix:

$$A_h^D \boldsymbol{u} = \boldsymbol{f} \text{ where } A_h^D = T_h^D + cI_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}, \tag{2.3}$$

where $\boldsymbol{u} = (u_1, \ldots, u_{n-1})^T$, $\boldsymbol{f} = (f_1, \ldots, f_{n-1})^T$, $I_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the identity matrix, and

$$T_h^D = \frac{1}{h^2} \operatorname{diag}(-1, 2, -1) \in \mathbb{R}^{(n-1)\times(n-1)}. \tag{2.4}$$

This is the simple, but very important, one dimensional case. In the case of higher spatial dimensions and on a uniform grid with the same step size $h = 1/n$ in all the directions the linear systems are written in compact form by using the standard tensor product $\otimes$ for matrices. We recall the following properties of the tensor product

$$(X + Y) \otimes Z = (X \otimes Z) + (Y \otimes Z), \quad (X_1 \otimes X_2)(Y_1 \otimes Y_2) = (X_1Y_1 \otimes X_2Y_2). \tag{2.5}$$

We further denote the k-th tensor power of a matrix X by $X^{\otimes k} = \underbrace{X \otimes \ldots \otimes X}_{k}$. Finally, let us note that the generalization to different step sizes in different directions is straightforward.

With this notation, the standard second order central difference scheme for discretization of the Dirichlet problem (2.1) results in the linear system

$$A_h^D \boldsymbol{u} = \boldsymbol{f}, \quad A_h^D = \sum_{j=1}^{d} \left(I_{n-1}^{\otimes(j-1)} \otimes T_h^D \otimes I_{n-1}^{\otimes(d-j)} \right) + cI_{n-1}^{\otimes d} \in \mathbb{R}^{(n-1)^d \times (n-1)^d}. \tag{2.6}$$