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Parameterized GSOR Method for a Class of Complex Symmetric Systems of Linear Equations

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Abstract. A parameterized generalized successive overrelaxation (PGSOR) method for a class of block two-by-two linear system is established in this paper. The convergence theorem of the method is proved under suitable assumptions on iteration parameters. Besides, we obtain a functional equation between the parameters and the eigenvalues of the iteration matrix for this method. Furthermore, an accelerated variant of the PGSOR (APGSOR) method is also presented in order to raise the convergence rate. Finally, numerical experiments are carried out to confirm the theoretical analysis as well as the feasibility and the efficiency of the PGSOR method and its variant.

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1 Introduction

Consider the system of linear equations

$$Au=b, \qquad A\in\mathbb{C}^{n\times n},\ u,b\in\mathbb{C}^n, \tag{1.1}$$

where A is a complex symmetric matrix with the form

$$A=W+iT,\quad (i=\sqrt{-1}), \tag{1.2}$$

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and $W,T \in \mathbb{R}^{n\times n}$ are both symmetric matrices with at least one of them being positive definite. Hereafter, without loss of generality, we assume that W is symmetric positive definite.

It is known to all that the Equation (1.1) arises frequently in many scientific and engineering applications. For instance, it comes from diffuse optimal tomography [1], molecular scattering [2], wave propagation [3], structural dynamics [4], FFT-based solution of certain time-dependent PDEs [5] and so on. More examples and additional practical backgrounds can be found in other places. See, e.g., [6] and references therein.

During these years, many methods with iteration approaches have been proposed for significantly approximating the unique solution of this system (1.1). For example, some of the well-known preconditioned Krylov subspace methods [5,7,8], Hermitian and skew-Hermitian splitting (HSS) method and lots of its variants [9–15], and C-to-R iteration methods [16–19] are proven to be useful techniques for solving the symmetric linear systems. As a matter of fact, our system (1.1) can be easily changed into a special case of generalized saddle point problems, so several generalizations of classical methods, such as generalized successive overrelaxation (GSOR) method in [20], the new variations of the method in [21–23] have brought new insight and new tools for solving such systems.

Based on GSOR method and some of its generalization, we in this paper develop a parameterized generalized SOR (PGSOR) method for solving the complex symmetric linear system (1.1). We study also the convergence properties and some its variants.

The organization of the paper is as follows. In Section 2, the new method PGSOR is established. In Section 3, the convergence analysis of the PGSOR method is exactly considered based on some lemmas. Section 4 will construct an efficient preconditioner to accelerate the convergence of the PGSOR method. In Section 5, some numerical examples are presented to show good behaviors for the efficiency of our methods. Finally, the paper is concluded in Section 6.

2 Parameterized GSOR method

Let $u=x+iy$ and $b=p+iq$, where $x,y,p,q\in\mathbb{R}^n$. Then the system (1.1) becomes

$$(W+iT)(x+iy)=p+iq. \tag{2.1}$$

And the complex linear system can be equivalently written as a block two-by-two real equivalent formulation

$$\mathcal{A}\widetilde{u}:=\begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} p \\ q \end{pmatrix}=:\widetilde{b}. \tag{2.2}$$

Inasmuch W is symmetric positive definite and T is symmetric, then the coefficient matrix $\mathcal{A}$ is nonsingular. A natural splitting of $\mathcal{A}$ is as follow,

$$\mathcal{A}=\mathcal{D}-\mathcal{L}-\mathcal{U},$$