On Multivariate Fractional Taylor’s and Cauchy’ Mean Value Theorem

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Abstract. In this paper, a generalized multivariate fractional Taylor’s and Cauchy’s mean value theorem of the kind

\[
\frac{f(x, y) - \sum_{j=0}^{n} \frac{D^j \alpha f(x_0, y_0)}{\Gamma(j \alpha + 1)}}{g(x, y) - \sum_{j=0}^{n} \frac{D^j \alpha g(x_0, y_0)}{\Gamma(j \alpha + 1)}} = \frac{R^\alpha_n(\xi, \eta)}{T^\alpha_n(\xi, \eta)},
\]

where \(0 < \alpha \leq 1\), is established. Such expression is precisely the classical Taylor’s and Cauchy’s mean value theorem in the particular case \(\alpha = 1\). In addition, detailed expressions for \(R^\alpha_n(\xi, \eta)\) and \(T^\alpha_n(\xi, \eta)\) involving the sequential Caputo fractional derivative are also given.

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1 Introduction

The ordinary Taylor’s formula has been generalized by many authors. Riemann [1] had already written a formal version of the generalized Taylor’s series:

\[
f(x+h) = \sum_{m=-\infty}^{\infty} \frac{h^{m+r}}{\Gamma(m+r+1)} (D_\alpha^{-(m+r)} f)(x),
\]

where \(D_\alpha^{-(m+r)}\) is the Riemann-Liouville fractional integral of order \(m+r\).

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Afterwards, Watanable [2] obtained the following relation:

\[
f(x) = \sum_{k=-m}^{n-1} \frac{(x-x_0)^{\alpha+k}}{\Gamma(\alpha+k+1)} (D^m \alpha f)(x_0) + R_{n,m},
\]

with \( m < \alpha, a \leq x_0 < x \), and

\[
R_{n,m} = (D^m_x (a+n) D^a f)(x) + \frac{1}{\Gamma(-\alpha-m)} \int_x^{x_0} (x-t)^{-\alpha-m-1} (D^a f(t))dt,
\]

where \( D^a f \) is the Riemann-Liouville fractional derivative of order \( a + n \).

On the other hand, a variant of the generalized Taylor’s series was given by Dzherbashyan and Nersesyan [3]. For \( f \) having all of the required continuous derivatives, they obtained

\[
f(x) = \sum_{k=0}^{m-1} \frac{(D^a f)(0)}{\Gamma(1+\alpha_k)} x^{\alpha_k} + \frac{1}{\Gamma(1+\alpha_m)} \int_0^x (x-t)^{\alpha_m-1} (D^a f(t))dt,
\]

where \( 0 < x, \alpha_0, \alpha_1, \ldots, \alpha_m \) is an increasing sequence of real numbers such that \( 0 < \alpha_k - \alpha_{k-1} \leq 1, k = 1, \ldots, m \) and \( D^a f = I_0^{1-\alpha_m} D_0^{1+\alpha_{m-1}} f \).

Under certain conditions for \( f \) and \( \alpha \in [0,1] \), Trujillo et al. [4] introduce the following generalized Taylor’s mean value theorem:

\[
f(x) = \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)a-1}}{\Gamma((j+1)a)} + R_n(f; \xi),
\]

\[
R_n(f; \xi) = \frac{(D^a \alpha f)(\xi)}{\Gamma((n+1)a+1)} (x-a)^{(n+1)a}, \quad a \leq \xi \leq x,
\]

\[
c_j = \Gamma(a) (x-a)^{1-\alpha} D^\alpha f(a+), \quad j = 0, 1, \ldots, n
\]

and the sequential fractional Riemann-Liouville derivative is denoted by

\[
D^{n\alpha}_a = D^\alpha_a \cdot D^\alpha_a \cdots D^\alpha_a (n \text{-times}).
\]

Recently, Odibat and Shawagfeh [5] obtain a new generalized Taylor’s mean value theorem of this kind

\[
f(x) = \sum_{j=0}^{n} \frac{(x-a)^{ja}}{\Gamma(ja+1)} (D^a \alpha f)(a) + \frac{(D^a \alpha f)(\xi)}{\Gamma((n+1)a+1)} (x-a)^{(n+1)a}
\]

with \( a \leq \xi \leq x \), where \( D^a \alpha \) is the sequential fractional Caputo derivative.

In 2005, Pecaric et al. [6] deduced the Cauchy type mean value theorem for the sequence fractional Riemann-Liouville derivative from known mean value theorem of the Lagrange type.