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A Multistep Scheme for Decoupled Forward-Backward Stochastic Differential Equations

Weidong Zhao[1,*], Wei Zhang[2,1] and Lili Ju[3]

[1] *School of Mathematics, Shandong University, Jinan, Shandong 250100, P. R. China.*

[2] *College of Applied Sciences, Beijing University of Technology, Beijing, 100022, P. R. China.*

[3] *Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.*

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Abstract. Upon a set of backward orthogonal polynomials, we propose a novel multi-step numerical scheme for solving the decoupled forward-backward stochastic differential equations (FBSDEs). Under Lipschtiz conditions on the coefficients of the FBSDEs, we first get a general error estimate result which implies zero-stability of the proposed scheme, and then we further prove that the convergence rate of the scheme can be of high order for Markovian FBSDEs. Some numerical experiments are presented to demonstrate the accuracy of the proposed multi-step scheme and to numerically verify the theoretical results.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Decoupled forward-backward stochastic differential equations, backward orthogonal polynomials, multi-step numerical scheme, error estimate, numerical analysis.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete, filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ ($T > 0$ a finite time) satisfying the usual conditions and $\mathcal{F}_0$ containing all the P-null sets of $\mathcal{F}$. On the probability space, a standard d-dimensional Brownian motion W_t is defined. Let $L^2 = L^2_{\mathcal{F}}(0,T)$ be the set of all $\mathcal{F}_t$-adapted and mean-square-integrable vector/matrix processes. We consider the system of decoupled forward-backward stochastic differential equations (FBSDEs)

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0,T], \tag{1.1}$$

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{1.2}$$

*Corresponding author. *Email addresses:* `wdzhao@sdu.edu.cn` (W. Zhao), `weizhang0313@bjut.edu.cn` (W. Zhang), `ju@math.sc.edu` (L. Ju)

where (1.1) is a forward stochastic differential equation (SDE) and (1.2) is a backward stochastic differential equation (BSDE). Assume that the coefficients b: $\Omega\times[0,T]\times\mathbb{R}^q\rightarrow\mathbb{R}^q$ and σ: $\Omega\times[0,T]\times\mathbb{R}^q\rightarrow\mathbb{R}^{q\times d}$, the generator f: $\Omega\times[0,T]\times\mathbb{R}^q\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\rightarrow\mathbb{R}^m$, and the terminal condition ξ is mean-square integrable and $\mathcal{F}_T$-measurable. We note that the integrals with respect to W_s in (1.1) and (1.2) are the Itô-type integrals. A process $(X_t,Y_t,Z_t):[0,T]\times\Omega\rightarrow\mathbb{R}^q\times\mathbb{R}^m\times\mathbb{R}^{m\times d}$ is called an L^2 solution of the decoupled FBSDEs (1.1) and (1.2) if, in the probability space $(\Omega,\mathcal{F},\mathbb{F},P)$, it is $\{\mathcal{F}_t\}$-adapted, square integrable and satisfies the Eqs. (1.1) and (1.2) [25]. In this paper, we assume that the system of decoupled FBSDEs (1.1) and (1.2) has a unique solution (X_t,Y_t,Z_t).

Under some standard conditions on the coefficients b and σ of the SDE (1.1) and the generator f of the BSDE (1.2), such as Lipschtiz conditions on b, σ and f, Pardoux and Peng first proved the existence and uniqueness of the solution of nonlinear BSDEs in [25] in 1990. Since then the theory of BSDEs and FBSDEs has been extensively studied, and important applications of them have been found in many fields, such as finance, risk measure, stochastic control, and so on [11, 13, 26–29]. Consequently it has become very important to solve the FBSDEs analytically or numerically for practical purposes. On the other hand it is often difficult to obtain analytic solutions of FBSDEs in explicitly closed form, or it is too complicated to compute the values of the solutions even we have the closed form of the solutions. Thus numerical methods for solving FBSDEs are highly desired, especially the methods with high efficiency and accuracy. There exist quite a few numerical methods for solving BSDEs and FBSDEs. Some of them are derived from the relationship between the FBSDEs and the corresponding parabolic PDEs, such as in [9, 17, 18, 21–23, 34], and others are obtained directly from discretizing BSDEs or FBSDEs, such as in [1–3, 5–8, 10, 12, 16, 19, 24, 29, 32, 33, 35–40]. Most of these algorithms are Euler-type methods with half order convergence rate [3, 4, 8–10, 17, 18, 21, 34]. In [12], a method with weak first order convergence was studied. In [16, 32, 35–40], some methods with convergence rates up to two were proposed and studied.

A multi-step scheme was first proposed for solving the BSDEs only (not combined together with forward SDEs). The high order of the scheme was numerically demonstrated and was theoretically proved for BSDEs with the generator f independent of Z_t. In [5, 6], the authors introduced high-order multi-step schemes for FBSDEs with general generators. They got the high-order error estimates for their schemes, in which the forward SDEs were not discretized.

In this paper, we propose a novel multi-step scheme for solving the decoupled FBSDEs (1.1) and (1.2). We first introduce a new set of orthogonal polynomials, which we call backward orthogonal polynomials and study some of their simple properties. Based on the theory of numerical integrals and polynomial approximation, we develop the new multi-step scheme by using the backward orthogonal polynomials and a special Gaussian process. Under not strong conditions, such as Lipschitz conditions on the coefficients b and σ of SDE (1.1) and the generator f of BSDE (1.2), we obtain error estimate of the proposed multi-step scheme in a very general form, which also implies