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High Order Energy-Preserving Method of the "Good" Boussinesq Equation

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Abstract. The fourth order average vector field (AVF) method is applied to solve the "Good" Boussinesq equation. The semi-discrete system of the "good" Boussinesq equation obtained by the pseudo-spectral method in spatial variable, which is a classical finite dimensional Hamiltonian system, is discretizated by the fourth order average vector field method. Thus, a new high order energy conservation scheme of the "good" Boussinesq equation is obtained. Numerical experiments confirm that the new high order scheme can preserve the discrete energy of the "good" Boussinesq equation exactly and simulate evolution of different solitary waves well.

AMS subject classifications: 65D17

Key words: "Good" Boussinesq equation, average vector field method, solitary waves, conservation law.

1. Introduction

The "good" Boussinesq (GB) equation provides a balance between dispersion and nonlinearity, which leads to the existence of soliton solutions, similar to the Korteweg-de Vries (KdV) equation and cubic nonlinear Schrödinger equation [1,19]. It describes shallow water waves propagating in both directions and possesses a highly complicated mechanism of solitary waves interaction and differs from other nonlinear wave equations. The solitary waves exist only for a finite range of velocities, they can merge into a single soliton, and they interact with each other to give rise to the so-called anti-solitons [6,12,13,15] and the references therein. The general form of the GB equation can be written as

$$u_{tt} - u_{xx} + u_{xxxx} - (u^2)_{xx} = 0, \tag{1.1}$$

in the region D $= \{(x,t) \in \mathbb{R}^2 : -L/2 \le x \le L/2, t \ge 0\}$, subject to the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{1.2}$$

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and the boundary conditions

$$u(-L/2,t)=0,\quad u(L/2,t)=0. \tag{1.3}$$

GB equation (1.1) possess the following two global conservation laws with the boundary conditions (1.3), namely global momentum conservation law

$$\mathcal{M}(t)=\int (vu_x)dx=\mathcal{M}(0), \tag{1.4}$$

and energy conservation law

$$\mathcal{E}(t)=\frac{1}{2}\int\Big(v^2+u^2+\frac{2}{3}u^3+u_x^2\Big)dx=\mathcal{E}(0), \tag{1.5}$$

where $v_x=u_t$.

Numerous numerical methods have been proposed to solve the GB equation (1.1). Frutos *et al.* [6]. developed the pseudo-spectral method of the GB equation; Soliton and anti-soliton interactions were studied by Manoranjan using the Galerkin-Petrov method [12, 13]; Ortega and Sanz-Serna [15] investigated the nonlinear stability and convergence behavior of numerical solutions; E-Zoheiry [20] studied the solitary wave interactions of the GB equation using finite-difference schemes; Huang *et al.* [7] constructed the multi-symplectic scheme of the GB equation; Aydin and Karasözen [2] constructed second order symplectic and multi-symplectic integrators for the GB equation using the two-stage Lobatto IIIA-IIIB partitioned Runge-Kutta method; Hu and Deng [8] proposed the implicit multi-symplectic scheme of the generalized Boussinesq equation. Chen [4] investigated the multi-symplectic Fourier pseudo-spectral method of the GB equation. Zeng [21] developed a new fifteen-point difference scheme which is equivalent to the multi-symplectic Preissman integrator.

Hamiltonian system, which has the energy conservation property, is one of the most important dynamical system. It is applied widely in the structural biology, pharmacology, semi-conductor, super-conducting, plasma, celestial mechanics, material, and partial differential equation, and so on. Feng and his research group [11, 16] developed symplectic geometric algorithms of the Hamiltonian system. Bridges and Reich developed the symplectic geometric algorithms to the multi-symplectic geometric algorithms of the partial differential equations [3]. Symplectic and multi-symplectic geometric algorithms [3,9,11,16,18], which have a long accurately computing capability, have been used widely in astronomy, molecular mechanics and quantum mechanics, electromagnetism, optics and so on. However, the symplectic and multisymplectic method only approximately preserve the energy of the Hamiltonian system (they exactly preserve a modified Hamiltonian) [9].

Recently, Quispel *et al.* [17] and McLachlan *et al.* [14] proposed the second order averaged vector field (AVF) method, which can preserve the energy of the Hamiltonian