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Fitted Mesh Method for a Class of Singularly Perturbed Differential-Difference Equations

Devendra Kumar*

Department of Mathematics, Birla Institute of Technology & Science, Pilani, Rajasthan-333031, India.

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Abstract. This paper deals with a more general class of singularly perturbed boundary value problem for a differential-difference equations with small shifts. In particular, the numerical study for the problems where second order derivative is multiplied by a small parameter ε and the shifts depend on the small parameter ε has been considered. The fitted-mesh technique is employed to generate a piecewise-uniform mesh, condensed in the neighborhood of the boundary layer. The cubic B-spline basis functions with fitted-mesh are considered in the procedure which yield a tridiagonal system which can be solved efficiently by using any well-known algorithm. The stability and parameter-uniform convergence analysis of the proposed method have been discussed. The method has been shown to have almost second-order parameter-uniform convergence. The effect of small parameters on the boundary layer has also been discussed. To demonstrate the performance of the proposed scheme, several numerical experiments have been carried out.

AMS subject classifications: 34K10; 34K28; 65D05

Key words: Singular perturbation, differential-difference equations, fitted-mesh, B-spline collocation method, boundary layer.

1. Introduction

In this paper, we consider the numerical approximation of the more general singularly perturbed differential-difference equation with small delay as well as advance with layer behavior. DDEs of this type arise naturally in the theoretical analysis of neuronal variability. There have been many advanced models of nerve membrane potential in the presence of random synaptic input. Reviews can be found in Fienberg [9], Holden [13], Segundo *et al.* [32]. In 1965, Stein [36] has given a differential-difference equation model incorporating stochastic effects due to neuronal variability and approximate the solution using Monte Carlo techniques. Stein's model contains the following assumptions:

*Corresponding author. *Email address:* dkumar2845@gmail.com (Devendra Kumar)

- Excitatory impulses arrive according to a Poisson process $\pi(f_e, t)$, each event of which leads to an instantaneous increase in the membrane depolarization $V(t)$ by a_e, whereas inhibitory current impulses arrive at event times in a second Poisson process $\pi(f_i, t)$, which is independent of $\pi(f_e, t)$ and causes $V(t)$ to decrease by a_i .
- If depolarization reaches a threshold of r units, the neuron fires an impulse.
- After each neuronal firing there is a refractory period of duration t_0, during which the impulses have no effect and the membrane depolarization $V(t)$, is reset to zero.
- At times $t > t_0$, each impulse produces unit depolarization.
- For sub-threshold levels, the depolarization decays exponentially among impulses with time constant μ.

This model and its modifications have been used as a basis for many studies devoted to the theoretical description of neuronal activities [3, 34, 37, 40]. One of the principle difficulty with the application of this model lies in solving the delay-differential equations that form the mathematical expression of the model [40]. Though there have been extensive studies of the properties of the solutions of many kinds of functional equations [2, 11] a little progress has been made on equations of type (1.1) with both forward (advance) and backward (delay) delays. These applications motivates the approximation of DDEs of Stein's model type.

In 1967, Stein [35] generalized this model to deal with a distribution of postsynaptic potential amplitudes. Johannesma [14] and Tuckwell [37] included the reversal potentials into account. Various other models for neuronal activity have been proposed and many are discussed in Holden's book [13].

We state a model problem for a general boundary value problem for a singularly perturbed differential-difference equation containing both type of shifts (negative as well as positive shifts)

$$\varepsilon^2 y''(x) + p(x)y'(x) + q(x)y(x-\delta) + r(x)y(x) + s(x)y(x+\eta) = f(x), \tag{1.1}$$

$\forall\, x \in \Omega = (0, 1)$ and subject to the interval conditions

$$y(x) = \phi(x) \quad \text{for } -\delta(\varepsilon) \le x \le 0, \quad y(x) = \psi(x) \quad \text{for } 1 \le x \le 1 + \eta(\varepsilon), \tag{1.2}$$

where ε is a small parameter $0 < \varepsilon \ll 1$, $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are of $o(\varepsilon)$. The functions $p(x)$, $q(x)$, $r(x)$, $s(x)$, $f(x)$, $\phi(x)$ and $\psi(x)$ are sufficiently smooth. For a function $y(x)$ to be smooth solution of problem (1.1)–(1.2), it must be continuous on $[0, 1]$ and continuously differentiable on (0, 1) satisfying (1.1). It is well-known [5, 27] for $\delta(\varepsilon) = \eta(\varepsilon) = 0$, the solution of the boundary value problem of the DDEs (1.1)–(1.2) exhibits layer at the left end or right end of the interval depending on the sign of $p(x)$.