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Interpolation, Projection and Hierarchical Bases in Discontinuous Galerkin Methods

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Abstract. The paper presents results on piecewise polynomial approximations of tensor product type in Sobolev-Slobodecki spaces by various interpolation and projection techniques, on error estimates for quadrature rules and projection operators based on hierarchical bases, and on inverse inequalities.

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1. Introduction

The topics covered in this paper belong to the fundamentals of the analysis of discontinuous Galerkin methods for partial differential equations. On the one hand, we have collected and reproved results from areas that are important for any study of finite element methods, such as the piecewise polynomial approximation in Sobolev spaces, quadrature formulas and inverse inequalities (Sections 2, 3, 4, 6). On the other hand, we have directed our attention to facts that are specifically related to particular techniques such as certain relations between lumping and quadrature effects or the investigation of fluctuation operators and shock-capturing terms by means of a hierarchical basis approach (Sections 4, 5). A major concern of our study was to trace the dependence of the constants on both the mesh width and the local polynomial degree.

The paper is organized as follows. After a brief introduction, which introduces the basic notation, we investigate polynomial approximations by means of tensor product elements on affine partitions in Section 3. This includes estimates of the reference transformations, which we prove by the help of a general chain rule. In this way we get a certain insight into the structure of the occurring constants. After that we prove error estimates for the Lagrange interpolation and the L_2-projection both with respect to

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the elements and with respect to the element edges in the scale of Sobolev-Slobodecki spaces.

In Section 4, we prepare some important notions such as quadrature formulas, lumping operators and discrete L_2-projections for later purposes. In particular, we point out the importance of an suitable choice of the quadrature points for optimal (w.r.t. the local polynomial degree) error estimates of the Lagrange interpolation.

In the following section we investigate the projection and interpolation errors for Gauss-Lobatto nodes. For this purpose we extend the concept of the hierarchical modal basis to the so-called *embedded hierarchical nodal basis* and we prove error estimates for the Lagrange interpolation and for the discrete L_2-projection which are optimal on the elements and almost optimal on the element edges. As a natural complement to the (direct) estimates from the previous sections, we present in Section 6 inverse inequalities that are based on generalizations of the Nikolski and Markov inequalities.

Beside Lemma 4.1 in Section 4, the key results are primarily contained in Section 5. However, a lot of other results is original in the sense that we were able to specify the constants (more precisely, upper bounds of them) that occur in the estimates in particular of Lemmas 3.1, 6.1, 6.4 and Corollary 6.1. For example, the knowledge of how the constants depend on the polynomial degree is important in the application to the convergence theory for finite element methods of higher accuracy. Moreover we tried to formulate the results, on the one hand, for a reasonably wide set of parameters, and, on the other hand, not only for the volume elements but also for their faces. We have also included some results (partially with sketched proofs) that are not reliably found in the literature.

2. Basic notation and definitions

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded polyhedral domain with a Lipschitzian boundary (see, e.g., [1, Def. 4.9]). Ω is subdivided by partitions $\mathcal{T}$ (in the sense of [11, Def. 1.49]) consisting of tensor product elements T (closed as subsets of $\mathbb{R}^d$) with diameter $h_T := \max_{x,y\in T} \|x-y\|_{\ell^2}$. Here and in what follows the symbol $\|\cdot\|_{\ell^p}$, $p \in [1,\infty]$, denotes the usual ℓ^p-norm of (finite) real sequences. Furthermore the maximal width of a partition $\mathcal{T}$ is defined by $h := \max\{h_T : \; T \in \mathcal{T}\}$. To indicate that a particular partition has the maximal width h, we will write $\mathcal{T}_h$.

In this paper, the standard definition of finite elements $\{T, P_T, \Sigma_T\}$ is used, see e.g. [11, Def. 1.23]. The finite element space is defined by

$$W_h := \left\{ w \in L^2(\Omega) : \; w|_T \in P_T \quad \forall T \in \mathcal{T}_h \right\},$$

where

$$P_T \subset W^{l,\infty}(T) \quad \forall T \in \mathcal{T}_h \quad \text{for some} \quad l \geq 0.$$

Here and in what follows, the symbol $W^{l,p}(T)$ denotes the standard Sobolev spaces of the (possibly fractional) degree of smoothness $l \geq 0$ and of the degree of summability