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Review of Methods Inspired by Algebraic-Multigrid for Data and Image Analysis Applications

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Abstract. Algebraic Multigrid (AMG) methods were developed originally for numerically solving Partial Differential Equations (PDE), not necessarily on structured grids. In the last two decades solvers inspired by the AMG approach, were developed for non PDE problems, including data and image analysis problems, such as clustering, segmentation, quantization and others. These solvers share a common principle in that there is a crosstalk between fine and coarse representations of the problems, with flow of information in both directions, fine-to-coarse and coarse-to-fine. This paper surveys some of these problems and the AMG-inspired algorithms for their solution.

AMS subject classifications: 65F10, 65F50, 65H10, 68U10
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1. Introduction

Multigrid methods were introduced in the 1960s and developed extensively in the 1970s [11], [12], [28], [29]. Originally devised for elliptic boundary value problems on structured grids, variants were introduced during these early years that were adapted to handling more and more complicated problems, including nonlinear problems, constraints, discontinuous coefficients and eigenvalue problems. The early 1980s saw the development of a new multigrid approach with far-reaching implications, namely, Algebraic Multigrid (AMG) ([13], [56], [50], [22], [14]). With this approach, the solvers were no longer restricted to problems defined on structured grids, nor necessarily discretized PDEs.

The fundamental idea of AMG was to choose a basic iterative method, the relaxation, and then devise a coarse-grid correction process that will be effective for reducing all error that was not efficiently eliminated by the relaxation. To this end, a

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heuristic algorithm was then developed for selecting a subset of the variables to be designated as the coarse-grid variables, along with a suitable prolongation matrix, and a Galerkin procedure was employed to project the problem onto the subspace spanned by the coarse-grid variables. This was applied recursively, resulting in a complete multilevel algorithm, quite similar in spirit to the classical multigrid approach except that all operations were defined algebraically without the need to consider an underlying PDE or even a computational grid. In the time that followed, many AMG variants were developed for various problems, but the large majority of these still aimed at solving linear systems of equations [58], or, less often, eigenproblems [9], [39].

In more recent years, multilevel methods have been developed in an ad-hoc manner for problems that are quite different in nature from those for which classical AMG can be applied, and yet these methods have often been inspired by AMG. In this paper we survey such algorithms for data analysis problems, mostly associated with Machine Learning and Image and Signal Processing. We focus on the problems of image segmentation, clustering and quantization along with the closely related problem of Voronoi tessellations, Markov Random Field (MRF) energy minimization, and Multi-Dimensional Scaling (MDS). Common to all the methods we survey is the multilevel structure and the cross-talk between the different scales associated with the problems.

The problems are reviewed in Section 2. Section 3 begins with a brief description of the classical AMG algorithm, followed by a generic description of variational coarsening for general convex functionals, leading to the classical FAS nonlinear multigrid algorithm [11]. The methods inspired by AMG for data and image analysis applications are described in Section 4. First, a FAS-like multigrid solver for MDS is reviewed, together with an application. After this, we describe multilevel methods for scalar and vector quantization and centroidal Voronoi tessellations, moving on to a multiscale algorithm for MRF energy minimization and finally clustering and segmentation by multilevel weighted aggregation. Each method is accompanied with a pseudo-code algorithm.

2. Applications and goals

2.1. Multidimensional scaling (MDS)

Multidimensional scaling is a generic name for a family of algorithms that embed points in target metric space from approximate inter-point distances, measured in some other metric space. MDS is widely used in dimensionality reduction, data analysis and visualization applications such as representing complicated high-dimensional data structures by low-dimensional ones [8], [40].

Problem definition. Let $\triangle$ be a symmetric $n \times n$ matrix of geodesic distances δ_{ij} measured between n points of a Riemannian manifold. The goal is to find a set of points $x_1, \ldots x_n$ in $\mathbb{R}^m$, such that the embedding error is minimal. A functional commonly used in MDS literature is the stress function

$$s(X;\triangle,W)=\sum_{i<j} w_{ij}(d_{ij}(X)-\delta_{ij})^2, \tag{2.1}$$