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A Multigrid Solver based on Distributive Smoother and Residual Overweighting for Oseen Problems

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Abstract. An efficient multigrid solver for the Oseen problems discretized by Marker and Cell (MAC) scheme on staggered grid is developed in this paper. Least squares commutator distributive Gauss-Seidel (LSC-DGS) relaxation is generalized and developed for Oseen problems. Residual overweighting technique is applied to further improve the performance of the solver and a defect correction method is suggested to improve the accuracy of the discretization. Some numerical results are presented to demonstrate the efficiency and robustness of the proposed solver.

Key words: Navier-Stokes equations, LSC-DGS, multigrid.

1. Introduction

We consider multigrid (MG) methods for the following linearized steady-state incompressible Navier-Stokes (NS) equations (Oseen model) in two dimensions:

$$\begin{cases} -\mu\Delta \boldsymbol{u} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g}, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu = 1/Re$, with Re the Reynold number, $\boldsymbol{u} = (u,v)^t$ is the velocity, $\boldsymbol{a} = (a(x,y), b(x,y))^t$ is the flow function satisfying $\operatorname{div}\boldsymbol{a} = 0$, $\boldsymbol{g}$ is the boundary data, and $\boldsymbol{f} = (f_1, f_2)^t$ is the external force. This linearized model usually comes from using the Picard's iteration to solve the NS equation, see, e.g. [13] (Section 7.2.2).

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Spatial discretization of the Oseen model (1.1) using either finite element or finite difference method leads to a large-scale sparse saddle point system of the following matrix form

$$\begin{pmatrix} F & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}, \tag{1.2}$$

where $\boldsymbol{u}$ now denotes the discrete velocity, p denotes the discrete pressure, F is the discretization of $-\mu\Delta + (\boldsymbol{a} \cdot \nabla)$, B' is the discrete gradient, and B is the (negative) discrete divergence.

Much work has been done for developing efficient solvers for (1.2), especially efficient preconditioners for Krylov subspace methods based on the block matrix form, see, e.g. [1, 13] and references therein. Multigrid methods have also been considered, for example [6, 14, 16, 18, 19, 22, 24, 25, 33]. We are interested in efficient MG methods that are robust with respect to both the mesh size h and the Reynold number Re.

For low Reynold number flow, John et. [18, 19] use multiple discretizations which combines a higher order finite element discretization with a lower order finite element approximation as a coarse grid solver. In [14], Fuchs and Zhao considered the distributive Gauss-Seidel (DGS) smoother and have shown that MG method using the DGS smoother works for enclosed flows in three dimensions with low Re numbers.

For high Reynolds number, the Oseen model becomes convection dominated and development of robust MG methods becomes more and more challenging. Brandt and Yavneh [6] propose a MG solver combined with a DGS smoother for high-Reynolds incompressible entering flows. They use standard or narrow upwind schemes of first or second order to discretize the convection term. Similar to the DGS smoother for Stokes problem [5], a good pressure convection-diffusion operator which almost commutes with the divergence operator is constructed to design an efficient DGS smoother. Based on such smoother, Thomas, Diskin and Brandt [24] obtain textbook multigrid efficiency for a model problem of flow past a finite flat plate. However, in this work, the construction of the pressure convection-diffusion operator is done for special flows, essentially constant flows, and it is not easy to generalize such construction to general flows. In [33], Zhang develop a MG solver with a second order upwind scheme for the convection term. Vanka smoother [25] with under-relaxation is used which is not robust with respect to the Re number. The number of iterations of MG cycles increases dramatically when Re number increases, i.e. $5 \sim 300$ steps with Re number from the range of $100 \sim 5000$. In [16], Hamilton, Benzi, and Haber considered MG methods for the Marker-and-Cell (MAC) discretization using smoothers based on Hermition/skew-Hermitian (HSS) and augmented Lagrangian (AL) splittings. For steady state Oseen problem, the proposed MG methods show moderate degeneracy on the Reynolds number up to $Re = 2048$.

In this paper, we consider least-square commutator distributive Gauss-Seidel (LSC-DGS) relaxation for solving the Oseen equation discretized by the MAC discretization with a first order upwind scheme. Central difference stencils are used for both convection and diffusion operators. To stabilize the scheme, the viscosity μ is replaced