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Multigrid Methods with Newton-Gauss-Seidel Smoothing and Constraint Preserving Interpolation for Obstacle Problems

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Abstract. In this paper, we propose a multigrid algorithm based on the full approximate scheme for solving the membrane constrained obstacle problems and the minimal surface obstacle problems in the formulations of HJB equations. A Newton-Gauss-Seidel (NGS) method is used as smoother. A Galerkin coarse grid operator is proposed for the membrane constrained obstacle problem. Comparing with standard FAS with the direct discretization coarse grid operator, the FAS with the proposed operator converges faster. A special prolongation operator is used to interpolate functions accurately from the coarse grid to the fine grid at the boundary between the active and inactive sets. We will demonstrate the fast convergence of the proposed multigrid method for solving two model obstacle problems and compare the results with other multigrid methods.

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1. Introduction

The obstacle problem is to find the equilibrium position of an elastic membrane which is constrained to lie below and/or above some given obstacles. Due to the obstacle constraints, the problem is often posed as a constrained minimization problem [3, 19]. Since the contact location of the membrane and the obstacle is usually unknown, sometimes the obstacle problem is studied as free boundary problem [5]. Obstacle problem can also be formulated as elliptic variational inequalities [8], linear complementarity problems [2, 17], and Hamilton-Jacobi-Bellman (HJB) equations [10]. All

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these formulations result in a nonlinear problem. A finite difference or finite element discretization will yield a nonlinear system of discrete equations. This paper considers fast solvers and in particular multigrid methods for solving the nonlinear discrete equations.

Many methods have been introduced to solve elliptic variational inequality partial differential equations (PDEs). Projected relaxation methods are popular techniques to solve elliptic variational inequalities [6, 7]. They are known to be easy to implement and convergent. However, the drawback of this approach is that its convergence depends on the choice of the relaxation parameter and has a slow asymptotic convergence rate.

Various forms of preconditioned conjugate gradient (PCG) algorithms for solving nonlinear variational inequalities are presented in [16]. They are more efficient than the projected relaxation method in some cases. However, the rates of convergence still depend on the size of the problem. For problems with small grid sizes, PCG methods may not be very efficient.

A multigrid method is introduced in [8] to solve the finite difference discretized PDE for an obstacle problem. Two phases are used in this method. The aim of the first phase, in which a sequence of coarse grids is used, is to get a good initial guess for the iterative phase two. In the second phase, the problem is solved by a W-cycle multigrid method. A cutting function is applied after the coarse grid correction. The convergence is shown to be better than PCG. However, the application of the cutting function and the phase one may lead to more expensive computations overall.

Elliptic variational inequalities can be reformulated as linear complementarity problems. A multigrid method, namely, projected full approximate scheme (PFAS), is proposed in [2] to solve linear complementarity problems arising from free boundary problems. The multigrid method is based on the full approximate scheme (FAS), which is often used for solving nonlinear PDEs. The multigrid method is built on a generalization of the projected SOR. Two further algorithms based on PFAS are introduced: PFASMD and PFMG, both of which are faster than PFAS. These methods show better convergence rates than the method in [8].

In [17], a PFAS multigrid is applied to solve the American style option problem which is formulated as a linear complementarity problem. The American style option problem can be viewed as an obstacle type problem where the obstacle is given by the payoff function. An F-cycle multigrid method is applied and a Fourier analysis of a smoother is provided. A comparison between an F-cycle and a V-cycle for solving linear complementarity problems is shown. In general, F-cycles show faster convergence than V-cycles [17]. However, an F-cycle requires more computations in each iteration, and so it is relatively more expensive.

Elliptic variational inequalities can also be reformulated as Hamilton-Jacobi-Bellman (HJB) equations. A multigrid algorithm which involves an outer and an inner iteration is proposed in [10]. The active and inactive sets of all grid levels are computed and stored in the outer loop. A W-cycle FAS multigrid method is applied to solve a linearized PDE in the inner iteration. An iterative step similar to [8] is adopted to compute a good