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Local Fourier Analysis of Multigrid Methods with Polynomial Smoothers and Aggressive Coarsening

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Abstract. We focus on the study of multigrid methods with aggressive coarsening and polynomial smoothers for the solution of the linear systems corresponding to finite difference/element discretizations of the Laplace equation. Using local Fourier analysis we determine *automatically* the optimal values for the parameters involved in defining the polynomial smoothers and achieve fast convergence of cycles with aggressive coarsening. We also present numerical tests supporting the theoretical results and the heuristic ideas. The methods we introduce are highly parallelizable and efficient multigrid algorithms on structured and semi-structured grids in two and three spatial dimensions.

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1. Introduction

For emerging many-core parallel architectures it has been observed that visiting the coarser levels of a multilevel hierarchy leads to a loss in performance, as measured by the percentage of peak performance achieved by the multigrid solver on such architectures. Roughly speaking, on the finer levels, computing residuals and smoothing can achieve relatively high performance, whereas on the coarser levels the performance

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of multigrid degrades due to the fact that fewer of the active threads are needed for computation there. These observations motivate the further study and development of multigrid methods that apply more smoothing on the finer levels together with aggressive coarsening strategies.

The use of point-wise smoothers (e.g., Jacobi and Gauss Seidel) together with aggressive coarsening in a multigrid solver has been studied using local Fourier analysis (LFA) in [19, 20] for rectangular grids and [10] for triangular grids. In these works, it has been observed that using aggressive coarsening is less efficient in terms of the total number of floating point operations than a more gradual coarsening approach, since these standard smoothing iterations are not able to effectively reduce a sufficiently large subspace of the high frequency components of the error.

However, polynomial smoothers are well suited for aggressive coarsening approach since they can be constructed to achieve a preset convergence rate on a given subspace corresponding, for example, to a subinterval of the high frequency components of the error. As shown in [3, 8, 12, 17] by using a sufficiently large degree in the polynomial approximation it is possible to guarantee prescribed damping on a preset subinterval of high frequency components. These works contain important results and provide efficient algorithms by adjusting the polynomial degree for a given coarsening ratio.

The focus of our work is on determining precisely the parameters of the polynomial smoothers, such as intervals of approximation, damping factors for high frequencies, coarsening ratios which result in best possible convergence rate. This, of course is an ambitious goal, but for semi-structured triangular and also rectangular grids this can be done. Our idea is to use the local Fourier analysis (LFA) to automatically determine the smoother and coarsening parameters which result in best performance. As we show, LFA allows us to obtain quantitative estimates of the performance of multigrid methods with polynomial smoothers of arbitrary degree and aggressive coarsening. As shown in [3, 4] polynomial smoothers result in algorithms with high degree of parallelism and outperform algorithms based on more classical relaxation methods. This is an additional advantage of the algorithms studied here as well.

The paper is organized as follows. We review some basic facts about two-grid and multigrid iterations in Section 2. Next, in Section 3, we introduce the polynomial smoothers of interest — all based on Chebyshev polynomials arising as solutions to different minimization problems — (1) an appropriately shifted and scaled classical Chebyshev polynomial smoother, (2) the so called smoothed aggregation polynomial, used as a smoother in [8], or (3) the best polynomial approximation to x^{-1} which is proposed as a smoother in [12]. The local Fourier analysis for these polynomial smoothers, together with a two-grid LFA for aggressive coarsening are presented in Section 3.1, as are their extensions to triangular grids. Then, bounds on the smoothing factors for the polynomials are calculated in Section 4. In Section 4, we also show how to utilize the LFA results in choosing the optimal parameters for the corresponding polynomial smoother. In Section 5 we present numerical tests illustrating the findings in the previous sections and also we provide extension to triangular grids in Section 5.3. Finally we draw some conclusions in Section 6.