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# Restarted Full Orthogonalization Method with Deflation for Shifted Linear Systems

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**Abstract.** In this paper, we study shifted restated full orthogonalization method with deflation for simultaneously solving a number of shifted systems of linear equations. Theoretical analysis shows that with the deflation technique, the new residual of shifted restarted FOM is still collinear with each other. Hence, the new approach can solve the shifted systems simultaneously based on the same Krylov subspace. Numerical experiments show that the deflation technique can significantly improve the convergence performance of shifted restarted FOM.

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**Key words**: Shifted linear systems, full orthogonalization method, restarted Arnoldi process, deflation, collinear.

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## 1. Introduction

Given a large sparse nonsymmetric matrix $A \in \mathbb{R}^{n\times n}$ and the right-hand side $b \in \mathbb{R}^n$, we want to simultaneously solving a sequence of shifted systems as follow

$$(A-\sigma_i I)x = b,\ \sigma_i \in \mathbb{R}, \quad i = 1,\cdots,s, \tag{1.1}$$

where $I$ denotes the $n\times n$ identity matrix. Such shifted systems arise in many scientific and engineering fields, such as control theory, image restorations, structural dynamics, and QCD problems, and thus attract a number of attention of many researchers, see [5, 8, 12, 13]. Among all the systems, when $\sigma = 0$, the system $Ax = b$ is treated as the seed system.

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Krylov subspace methods, for instance, FOM and GMRES, are popular choices for solving large sparse system of linear equations, which construct an orthonormal basis of Krylov subspace of the form,

$$\mathcal{K}_m(A, v) = \mathrm{span}\{v, Av, \cdots, A^{m-1}v\}.$$

Since the storage requirement and computational cost of FOM and GMRES will become expensive as $m$ increases, restarting is often taken into use.

Let $x_0$ be an initial vector, in order to solve a sequence of shifted systems (1.1), a number of Krylov subspaces

$$\mathcal{K}_m(A-\sigma_i I, r_0^{(i)}) = \mathrm{span}\{r_0^{(i)}, (A-\sigma_i I)r_0^{(i)}, \cdots, (A-\sigma_i I)^{m-1}r_0^{(i)}\}, \quad i = 1, \cdots, s, \tag{1.2}$$

will be considered where $r_0^{(i)} = b - (A - \sigma_i I)x_0$ are the initial residuals, $i = 1, \cdots, s$.

It is shown in [3] that Krylov subspace is shift invariant, i.e.,

$$\mathcal{K}_m(A, v) = \mathcal{K}_m(A - \sigma I, v), \quad \forall\, \sigma. \tag{1.3}$$

It shows that the Krylov subspaces $\mathcal{K}_m(A - \sigma I, v)$ can be spanned by the same basis regardless of the choice of parameter $\sigma$, which shed a light that the shifted systems (1.1) can be solved simultaneously based on only one Krylov subspace.

From formula (1.2), it is seen that when the initial residual $r_0^{(i)}, i = 1, \cdots, s$, are parallel to each other, FOM and GMRES can be applied to solve (1.1) simultaneously at the first cycle. Usually, the initial vector for the next cycle is the current residual vector associated with an approximate solution. This means that all residual vectors formed at the end of a cycle are required to be collinear with each other, so that even in restarted scheme, the shifted systems (1.1) can always be solved based on the same Krylov subspace.

For GMRES, the collinearity of residuals are lost after the first restart, a variant of GMRES has been presented in [8] for (1.1) by forcing the residuals to be parallel. Note that in this case, only the base system has the minimum residual property, the solution of the other shifted systems is not equivalent to GMRES applied to those systems.

Note that the collinearity of residuals are satisfied automatically for FOM. Hence, Simoncini [5] gave an effective restarted FOM to solve all shifted linear systems simultaneously. Jing and Huang in [13] further accelerated this method by introducing a weighted norm.

In this paper, we study restarted FOM with the deflation technique for solving all shifted linear systems simultaneously. Due to Restarting generally slows the convergence of FOM by discarding some useful information at the restart, the deflation technique can compensate this disadvantage in some sense by keeping the Ritz vectors from the last cycle. This idea was first proposed by Morgan in [10] and then widely studied in [4, 9, 11].

Theoretical analysis show that the advantage of deflated restarted FOM is that the collinearity property of residual vectors by the deflated Arnoldi process are still maintained. Thus, the restart technique can be applied and finding the solution of the shifted