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Interpolation by G^2 Quintic Pythagorean-Hodograph Curves

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Abstract. In this paper, the G^2 interpolation by Pythagorean-hodograph (PH) quintic curves in $\mathbb{R}^d$, $d \geq 2$, is considered. The obtained results turn out as a useful tool in practical applications. Independently of the dimension d, they supply a G^2 quintic PH spline that locally interpolates two points, two tangent directions and two curvature vectors at these points. The interpolation problem considered is reduced to a system of two polynomial equations involving only tangent lengths of the interpolating curve as unknowns. Although several solutions might exist, the way to obtain the most promising one is suggested based on a thorough asymptotic analysis of the smooth data case. The numerical algorithm traces this solution from a particular set of data to the general case by a homotopy continuation method. Numerical examples confirm the efficiency of the proposed method.

AMS subject classifications: 65D05, 65D17

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1. Introduction

Polynomial Pythagorean-hodograph (PH) curves form a special subclass of parametric curves and have several important properties, such as a (piecewise) polynomial

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arc length, planar PH curves have rational offsets and spatial PH curves possess rational adapted frames. By definition, a parametric polynomial curve $\boldsymbol{p} : [0,1] \to \mathbb{R}^d$ of degree $\leq n$ is a PH curve if the Euclidean norm of its hodograph

$$\|\boldsymbol{p}'\| = \sqrt{\boldsymbol{p}'^T \boldsymbol{p}'}$$

is a piecewise polynomial of degree $\leq n-1$. They were first systematically studied in [10]. Since then, a lot of research work was put into studying them in detail.

A polynomial PH curve is usually obtained by the integration of the hodograph, constructed in a particular way from a so called preimage, which is a complex polynomial for a planar PH curve and a quaternion polynomial for a spatial PH curve (see e.g. [7, 8]). A generalization to higher dimensional curves involves tools from Clifford algebras [5, 22]. These constructions have many nice properties, specially from the computational point of view, but might not be so convenient when dealing with highly nonlinear geometric interpolation problems where the curve is completely determined only from geometric quantities like points, tangent directions and curvatures. Namely, to determine a geometric interpolant one needs to compute also the unknown parameters at which the points are interpolated, unknown tangent lengths, etc., and it is very desirable that these type of unknowns (geometric parameters) are separated from the unknown coefficients of the curve. When the curve is constructed from a complex or a quaternion polynomial, a separation of unknowns is difficult to achieve unless one increases the degree of the preimage which results in a PH curve of a higher degree [15, 23]. An alternative approach to determine a PH curve that interpolates the data in a geometric sense was proposed in [14], where a separation of geometric parameters from the coefficients of a curve is right at hand. Moreover, it provides a way to analyse the interpolation problem independently of the dimension of the space. This property is connected to the fact that a PH curve of degree n can interpolate $n+1$ geometric data regardless of the dimension, as was conjectured in [12].

Many interpolation and approximation methods using PH curves have been developed. Since only odd degree PH curves are regular, cubic and quintic curves are probably the most interesting from the practical point of view. Interpolation schemes using planar cubic PH curves can be found in [3, 12, 13, 19], and spatial PH cubics are considered in [14, 16, 17, 21]. For planar quintic PH curves, C^2 continuous splines that interpolate given points were constructed in [1] and G^1 Hermite interpolation problem was studied in [4]. Interpolation of G^1 Hermite data together with end curvatures by monotone helical quintics, which form a subclass of general spatial PH curves, is done in [11].

When designing curves it is often desirable to join two points with a G^2 contact. Not many results are known for this type of interpolation using PH curves of low degree. A planar cubic G^2 spline interpolating only the given points is constructed in [13]. A spatial G^2 continuous curve composed of a pair of PH quintic spiral segments is examined in [20]. For a G^2 interpolation scheme with PH curves of degree seven see [15].