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Product Gaussian Quadrature on Circular Lunes

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Abstract. Resorting to recent results on subperiodic trigonometric quadrature, we provide three product Gaussian quadrature formulas exact on algebraic polynomials of degree n on circular lunes. The first works on any lune, and has $n^2 + \mathcal{O}(n)$ cardinality. The other two have restrictions on the lune angular intervals, but their cardinality is $n^2/2 + \mathcal{O}(n)$.

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1. Introduction

Quadrature problems concerning a circular lune, i.e., the portion of a disk not "obscured" by an overlapping disk, have a very long and fascinating history, starting from the famous quadrature of Hippocrate's lunes (5th century BC), passing through Clausen's and Euler's contributions in the 18th century, till the classification of the lunes that are constructible by compass and straightedge and that have equal area to a given square, eventually obtained in the mid-20th century by the Russian mathematicians Chebotaryov and Dorodnov via Galois theory; cf., e.g., [11, 15].

In the present paper, we study the problem of *quadrature on lunes*, that is numerical integration of a bivariate function on a circular lune, constructing three product Gaussian quadrature formulas that are *exact on algebraic polynomials* up to a given degree. Quite surprisingly, such a problem has not been yet addressed in the numerical literature, at least in the present formulation concerning polynomial exactness on the original domain. Some approaches have indeed been studied, for example in [16, 17], where however a preliminary polynomial or spline approximation of the lune boundary is needed.

The key of our approach is given by suitable trigonometric transformations, that map (the interior of) a rectangle (in angular variables) diffeomorphically onto (the interior of) the lune, and allow to resort to some recently developed trigonometric

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Gaussian rules on subintervals of the period (cf. [5]). In such a way we complete our study of algebraic cubature on the geometrical figures related to a couple of overlapping disks: lenses (intersection of two disks), double bubbles (union of two disks), cf. [4], and lunes (difference of two disks).

All these results, indeed, and many other concerning sections of the disk (circular segments, sectors, zones) and more generally integration domains related to *circular and elliptical arcs*, are based on the recently developed topic of "subperiodic" trigonometric interpolation and quadrature; cf. [2,3,5,6,10].

For the reader's convenience, we report the main result of [5], stated here for a general angular interval:

Proposition 1.1. *Let* $[\alpha,\beta]$ *be an angular interval, with* $0 < \beta-\alpha \leq 2\pi$. *Let* $\{(\xi_j,\lambda_j)\}_{1\leq j\leq n+1}$, *be the nodes and positive weights of the algebraic Gaussian quadrature formula for the weight function*

$$w(x) = \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,x^2}}, \quad x\in(-1,1), \quad \omega = \frac{\beta-\alpha}{2} \leq \pi. \tag{1.1}$$

Then

$$\int_\alpha^\beta t(\theta)\,d\theta = \sum_{j=1}^{n+1} \lambda_j t(\theta_j), \tag{1.2}$$

for every trigonometric polynomial $t\in\mathbb{T}_n([\alpha,\beta])$, *where*

$$\theta_j = \frac{\alpha+\beta}{2} + 2\arcsin\left(\xi_j\,\sin\left(\frac{\omega}{2}\right)\right) \in (\alpha,\beta), \quad j=1,2,\cdots,n+1.$$

Observe that, since the weight function (1.1) is even, the set of angular nodes is symmetric with respect to the center of the interval, and that symmetric nodes have equal weight, cf. [8].

The paper is organized as follows. In Sections 2 and 3 we develop the theoretical construction of three product Gaussian quadrature formulas exact on algebraic polynomials of degree n on circular lunes. The first works on any lune, and has $n^2+\mathcal{O}(n)$ cardinality. The other two have restrictions on the lune angular intervals, but their cardinality is $n^2/2+\mathcal{O}(n)$. Moreover, we discuss the convergence rate of such formulas in connection with multivariate Jackson inequality.

In Section 4, we present some numerical results obtained by a Matlab implementation of the product quadrature formulas. The corresponding codes are available online in [7]. We do not develop specific applications here, but it is worth recalling that Gaussian quadrature on domains defined by circular arcs is of practical interest, for example, in the field of optical design and optimization; cf. [1].

2. A general transformation

In this section we construct a basic product quadrature formula, which is valid on any circular lune. By no loss of generality, up to rotation, translation and scaling, we