Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2014.1231nm

Vol. 7, No. 1, pp. 58-106
February 2014

pde2path – A Matlab Package for Continuation and Bifurcation in 2D Elliptic Systems

Hannes Uecker[1,*], Daniel Wetzel[1] and Jens D. M. Rademacher[2]

[1] *Institut für Mathematik, Universität Oldenburg, 26111 Oldenburg, Germany.*
[2] *Universität Bremen, Fachbereich Mathematik, Postfach 33 04 40, 28359 Bremen, Germany.*

Received 26 September 2012; Accepted (in revised version) 2 May 2013

Available online 24 January 2014

Abstract. `pde2path` is a free and easy to use `Matlab` continuation/bifurcation package for elliptic systems of PDEs with arbitrary many components, on general two dimensional domains, and with rather general boundary conditions. The package is based on the FEM of the `Matlab` `pdetoolbox`, and is explained by a number of examples, including Bratu's problem, the Schnakenberg model, Rayleigh-Bénard convection, and von Karman plate equations. These serve as templates to study new problems, for which the user has to provide, via `Matlab` function files, a description of the geometry, the boundary conditions, the coefficients of the PDE, and a rough initial guess of a solution. The basic algorithm is a one parameter arclength-continuation with optional bifurcation detection and branch-switching. Stability calculations, error control and mesh-handling, and some elementary time-integration for the associated parabolic problem are also supported. The continuation, branch-switching, plotting etc are performed via `Matlab` command-line function calls guided by the `AUTO` style. The software can be downloaded from `www.staff.uni-oldenburg.de/hannes.uecker/pde2path`, where also an online documentation of the software is provided such that in this paper we focus more on the mathematics and the example systems.

AMS subject classifications: 35J47, 35J60, 35B22, 65N30

Key words: Elliptic systems, continuation and bifurcation, finite element method.

1. Introduction

For algebraic systems, ordinary differential equations (ODEs), and partial differential equations (PDEs) in one spatial dimension there is a variety of software tools for the numerical continuation of families of equilibria and detection and following of bifurcations.

*Corresponding author. *Email addresses:* `hannes.uecker@uni-oldenburg.de` (H. Uecker), `daniel.wetzel@uni-oldenburg.de` (D. Wetzel), `rademach@math.uni-bremen.de` (J. Rademacher)

These include, e.g., `AUTO` [11], `XPPaut` [10] (which relies on `AUTO` for the continuation part) and `MatCont` [14], see also `www.enm.bris.ac.uk/staff/hinke/dss/` for a comprehensive though somewhat dated list. Another interesting approach is the "general continuation core" `coco`, [31].

However, for elliptic systems of PDEs with two spatial dimensions there appear to be few general continuation/bifurcation tools and hardly any that work out-of-the-box for non-expert users. PLTMG [2] treats scalar equations, and there are many case studies using ad hoc codes, often based on `AUTO` using suitable expansions for the second spatial direction; for 2D systems there also is `ENTWIFE` [38], which however appears to be no longer maintained since 2001. For experts we also mention `Loca` [29], which is designed for large scale problems, and `oomph` [16], another large package which also supports continuation/bifurcation, though this is not yet documented.

Our software `pde2path` is intended to fill this gap. Its main design goals and features are:

- **Flexibility and versatility.** The software is based on the `Matlab` `pdetoolbox` and treats PDE systems

$$G(u,\lambda) := -\nabla \cdot (c \otimes \nabla u) + au - b \otimes \nabla u - f = 0, \tag{1.1}$$

  where $u = u(x) \in \mathbb{R}^N$, $x \in \Omega \subset \mathbb{R}^2$ some bounded domain, $\lambda \in \mathbb{R}$ is a parameter, $c \in \mathbb{R}^{N\times N\times 2\times 2}$, $b \in \mathbb{R}^{N\times N\times 2}$ (see (1.4a), (1.4b) below), $a \in \mathbb{R}^{N\times N}$ and $f \in \mathbb{R}^N$ can depend on $x, u, \nabla u$, and, of course, parameters. The standard assumption is that c, a, f, b depend on $u, \nabla u, \cdots$, locally, e.g., $f(x,u) = f(x,u(x))$; however, the dependence of c, a, f, b on arguments *can* in fact be quite general, for instance involving global coupling, see Section 3.5. The current version supports "generalized Neumann" boundary conditions (BC) of the form

$$\mathbf{n} \cdot (c \otimes \nabla u) + qu = g, \tag{1.2}$$

  where $\mathbf{n}$ is the outer normal and again $q \in \mathbb{R}^{N\times N}$ and $g \in \mathbb{R}^N$ may depend on x, u, ∇u and parameters. These boundary conditions include zero flux BC, and a "stiff spring" approximation of Dirichlet BC via large prefactors in q and g, that we found to work well.

  There are a number of predefined functions to specify domains Ω and boundary conditions, or these can be exported from `matlab`'s pdetoolbox GUI, thus making it easy to deal with (almost) arbitrary geometry and boundary conditions. The software can also be used to time-integrate parabolic problems of the form

$$\partial_t u = -G(u,\lambda), \tag{1.3}$$

  with G as in (1.1). This is mainly intended to easily find initial conditions for continuation. Finally, any number of eigenvalues of the Jacobian $G_u(u,\lambda)$ can be computed, thus allowing stability inspection for stationary solutions of (1.3).