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The Numerical Simulation of Space-Time Variable Fractional Order Diffusion Equation

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Abstract. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The continuum of order in the fractional calculus allows the order of the fractional operator to be considered as a variable. Numerical methods and analysis of stability and convergence of numerical scheme for the variable fractional order partial differential equations are quite limited and difficult to derive. This motivates us to develop efficient numerical methods as well as stability and convergence of the implicit numerical methods for the space-time variable fractional order diffusion equation on a finite domain. It is worth mentioning that here we use the Coimbra-definition variable time fractional derivative which is more efficient from the numerical standpoint and is preferable for modeling dynamical systems. An implicit Euler approximation is proposed and then the stability and convergence of the numerical scheme are investigated. Finally, numerical examples are provided to show that the implicit Euler approximation is computationally efficient.

AMS subject classifications: 26A33, 34K28, 65M12, 60J70

Key words: Variable fractional derivative, diffusion equation, implicit Euler scheme, stability, convergence.

1. Introduction

The fractional diffusion equation (FDE) is a generalization of the classical diffusion equation by replacing the integer-order derivatives by fractional-order derivatives, which is a useful approach for the description of transport dynamics in complex systems governed by anomalous dispersion and non-exponential relaxation [1–3]. Recently, more and more researchers find that many dynamic processes appear to exhibit fractional order behavior that may vary with time or space, which indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex

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dynamical problems. For example, Variable-order has applied to viscoelasticity [4], the processing of geographical data [5], signature verification [6], diffusion [7], etc. While the variable-order definitions were studied in the 1990s, Samko and Ross [8] first discussed some properties and the inversion formula of the variable-order operator $(\frac{d}{dx})^{\alpha(x)}f(x)$ using the Riemann-Liouvile definition and Fourier transforms. Hereafter some mapping properties in Hölder spaces and "measure of deviation" of the direct generalized operators of the Riemann-Liouville fractional integration and differentiation and the Marchaud form to the case of variable order $\alpha(x)$ were considered by them [9, 10]. Kikuchi and Negoro [11] investigated the relationship between Markov processes and evolution equations with respect to pseudo differential operators. In 1998, Lorenzo and Hartley [12] suggested that the concept of variable-order (or order structure) operator is allowed to vary either as a function of the independent variable of integration or differentiation (t) or as a function of some other (perhaps spatial) variable (y). At the same time, a preliminary study was done in several potential variable-order definitions and initial properties were forwarded. Hereafter, in 2002, they [13] developed more deeply the concept of variable and distributed order fractional operators based on the Riemann-Liouville definition and other new operators, then the relationship between the mathematical concepts and physical processes were investigated. Afterward, a few researchers put different definitions of variable fractional order operators to suit desired goals and discussed their applications respectively [4, 7, 14, 15]. In the recent research article, Ramirez et al. [16] compared nine variable-order operator definitions based on a very simple criteria: the variable order operator must return the correct fractional derivative that corresponds to the argument of the functional order. They found that only Marchaud-definition and Coimbra-definition satisfied the above elementary requirement, and pointed Coimbra-definition variable-order operator was more efficient from the numerical standpoint and then preferable for modeling dynamics systems.

The research on variable-order fractional partial differential equations is relatively new, and numerical approximation of these equations is still at an early stage of development. Lin et al. [17] studied the stability and convergence of an explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. Zhuang et al. [18] discussed the stability and convergence of Euler approximation for the variable fractional order advection-diffusion equation with a nonlinear source term, moveover, they presented some other numerical methods for the equation. Chen et al. [19] considered a variable-order anomalous subdiffusion equation, in the paper, two numerical schemes were proposed, one with first order temporal accuracy and fourth order spatial accuracy, the other with second order temporal accuracy and fourth order spatial accuracy. Here, we point out that in above these papers, variable-order derivative is either space derivative or time derivative. According to the authors knowledge, there are not literatures consider numerical approximation of variable-order problem containing both space variable-order derivative and time variable-order derivative.

In the paper we consider the following space-time variable fractional order diffusion equation (STVFODE):