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A Source Transfer Domain Decomposition Method For Helmholtz Equations in Unbounded Domain Part II: Extensions

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Abstract. In this paper we extend the source transfer domain decomposition method (STDDM) introduced by the authors to solve the Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains. The STDDM is based on the decomposition of the domain into non-overlapping layers and the idea of source transfer which transfers the sources equivalently layer by layer so that the solution in the final layer can be solved using a PML method defined locally outside the last two layers. The details of STDDM is given for each extension. Numerical results are presented to demonstrate the efficiency of STDDM as a preconditioner for solving the discretization problem of the Helmholtz problems considered in the paper.

AMS subject classifications: 65F08, 65N55, 65Y20

Key words: Helmholtz equation, high frequency waves, PML, source transfer.

1. Introduction

The source transfer domain decomposition method (STDDM) is introduced by the authors in [11] to solve the following 2D Helmholtz problems:

$$\Delta u + k^2 u = f \qquad \text{in } \mathbb{R}^2, \tag{1.1a}$$

$$r^{\frac{1}{2}}\Big(\frac{\partial u}{\partial r} - \mathbf{i}ku\Big) \to 0 \qquad \text{as } r = |x| \to \infty, \tag{1.1b}$$

where $k > 0$ is the wave number and $f \in H^{-1}_{\rm comp}(\mathbb{R}^2)$, that is, $f \in H^{-1}(\mathbb{R}^2)$ and has compact support.

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Helmholtz equation (1.1a) appears in diverse scientific and engineering applications including acoustics, elasticity, and electromagnetics. It is well-known that the efficient algebraic solver for large wave number discrete Helmholtz equation resulting from finite difference or finite element discretization is challenging due to the huge number of degrees of freedom required and the highly indefinite nature of the discrete problem. There exist considerable efforts in the literature for finding efficient algorithms for solving discrete Helmholtz equations, see e.g., Benamou and Després [4], Gander et al. [15], Brandt and Livshit [6], Elman et al. [12], and the review articles Erlangga [14], Osei-Kuffuor and Saad [19] and the references therein. The STDDM is motivated by the recent work of Engquist and Ying [13] in which a sweeping preconditioner is constructed by an approximate LDL^t factorization which eliminates the unknowns layer by layer. The Schur complement matrix of the factorization is approximated by using a moving perfectly matched layer (PML) technique.

The purpose of this paper is to extend the STDDM to solve the Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains. Let $\Omega_i = \{x \in \mathbb{R}^2 : \zeta_i < x_2 < \zeta_{i+1}\}$, $i = 1, \cdots, N$, be the layers whose union covers the support of the source f. Let $\Omega_0 = \{x \in \mathbb{R}^2 : x_2 < \zeta_1\}$ and $\Omega_{N+1} = \{x \in \mathbb{R}^2 : x_2 > \zeta_{N+1}\}$. Let f_i be the restriction of f in Ω_i and vanish outside Ω_i. It is clear that

$$u(x) = -\int_{\mathbb{R}^2} f(y)G(x,y)dy = -\sum_{i=1}^{N}\int_{\Omega_i} f_i(y)G(x,y)dy, \quad G(x,y) = \frac{\mathbf{i}}{4}H_0^{(1)}(k|x-y|).$$

Let $\bar{f}_1 = f_1$. The key idea of STDDM is to define a source transfer operator Ψ_{i+1} that transfers the source from Ω_i to Ω_{i+1} in the sense that

$$\int_{\Omega_i} \bar{f}_i(y)G(x,y)dy = \int_{\Omega_{i+1}} \Psi_{i+1}(\bar{f}_i)(y)G(x,y)dy, \quad \forall x \in \Omega_j, \quad j > i+1. \tag{1.2}$$

Then for $\bar{f}_{i+1} = f_{i+1} + \Psi_{i+1}(\bar{f}_i)$ we have

$$u(x) = -\int_{\Omega_N} f_N(y)G(x,y)dy - \int_{\Omega_{N-1}} \bar{f}_{N-1}(y)G(x,y)dy, \quad \forall x \in \Omega_N. \tag{1.3}$$

The solution u in Ω_N only involves the sources in Ω_N and Ω_{N-1} and thus can be solved locally by using the PML method defined outside only two layers Ω_N and Ω_{N-1}. Once the solution u in Ω_N is known, the solution in the other layers can be computed successively by solving the half-space Helmholtz problem using the transferred sources. This heuristic idea is made rigorous in the setting of PML method in [11].

The layout of the paper is as follows. In Section 2 we will review the basic ingredients of the PML method and STDDM method for constant wave number. In Sections 3-5 we propose STDDM for solving Helmholtz problems in two-layered media, the Helmholtz scattering problems with bounded scatterer, and Helmholtz problems in 3D unbounded domains separately. The details of STDDM for each extension is provided and numerical examples are included to show the effective behavior of STDDM as a preconditioner.