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Spectral Optimization Methods for the Time Fractional Diffusion Inverse Problem

Xingyang Ye[1,2] and Chuanju Xu[1,*]

[1] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, Fujian, China.*
[2] *School of Science, Jimei University, Xiamen 361021, Fujian, China.*

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Abstract. An inverse problem of reconstructing the initial condition for a time fractional diffusion equation is investigated. On the basis of the optimal control framework, the uniqueness and first order necessary optimality condition of the minimizer for the objective functional are established, and a time-space spectral method is proposed to numerically solve the resulting minimization problem. The contribution of the paper is threefold: 1) a priori error estimate for the spectral approximation is derived; 2) a conjugate gradient optimization algorithm is designed to efficiently solve the inverse problem; 3) some numerical experiments are carried out to show that the proposed method is capable to find out the optimal initial condition, and that the convergence rate of the method is exponential if the optimal initial condition is smooth.

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Key words: Time fractional diffusion equation, inverse problem, spectral method, error estimate, conjugate gradient method.

1. Introduction

Optimal control problems can be found in many scientific and engineering applications, and it has become a very active and successful research area in recent years. Extensive research has been carried out on various theoretical aspects of control problems such as existence of optimal control, optimality conditions, regularity of the optimal solutions, and so on. The literature on this field is huge, and it is impossible to give even a very brief review here. However, to the best of the authors' knowledge, most research concerning control problems has been performed using partial differential equations of integer order, and there are not many published works related to the differential equations of fractional order.

*Corresponding author. *Email addresses:* xingyangye@163.com (X. Y. Ye), cjxu@xmu.edu.cn (C. Xu)

In this paper we are interested in control problems based on partial differential equations of fractional order. This work is motivated by the fact that the fractional partial differential equations are novel extensions of the traditional models, based on fractional calculus. They are now winning more and more scientific applications cross a variety of fields including control theory, biology, electrochemical processes, viscoelastic materials, polymer, finance, and etc. We will consider an inverse problem associated to the time fractional diffusion equation (TFDE) of the form

$$\partial_t^\alpha u(x,t) - \partial_x^2 u(x,t) = 0, \quad 0 < x < l, \quad t > 0,$$

where ∂_t^α means the fractional derivative of order α, $0 < \alpha < 1$. Precisely, the problem to be investigated is as follows: suppose we are given the observed date $\bar{u}(x,t)$, the goal is to find out the optimal initial condition $u(x,0)$ such that the corresponding solution to TFDE matches the observed data as closely as possible.

For the initial boundary value problem of TFDE, some theoretical and numerical results have been obtained by a number of authors. For example, Schneider and Wyss [18] and Wyss [20] used the Green functions to construct the explicit solution in some simple cases. Luchko [12, 13] derived the maximum principle and proved the unique existence of the generalized solution. Sakamoto and Yamamoto [16] investigated weak solutions of TFDE in 2D. The existing numerical methods includes finite difference [7, 11, 19], Galerkin finite element [4, 5, 15], finite difference/spectral method [9], time-space spectral method [8], and so on.

For the inverse problem concerning TFDE, although the research is relatively sparse, several studies have been carried out, and we see increasing interest in this topic from both scientific and engineering communities. We mention, among others, the work [3] to determine the fractional order α and variable diffusion coefficient by means of additional boundary data. The uniqueness of the inverse problem was proved theoretically on the basis of the eigenfunction expansion of the weak solution and the Gel'fand-Levitan theory. Sakamoto and Yamamoto also considered in their above mentioned paper an inverse source problem. They analyzed the stability of determining time-dependent factor in the source by some observation. Zhang and Xu [21] established the uniqueness of an inverse problem which consists in identifying the time independent source term for TFDE with homogenous Neumann boundary condition, and some numerical examples were presented.

In this work, we will focus on the numerical method to find out the optimal initial condition for the TFDE with known observed data. Unlike the work [21], which uses the eigenfunction expansion of the solution as the main tool, we will adapt the optimal control framework [10] to treat the inverse problem. By introducing an objective function which measures the discrepancy of the solution given by the TFDE problem and the known observation data, the optimal initial condition is then defined as the state such that the objective function attains its minimum. Thanks to the weak formulation of TFDE proposed in [8], we are able to derive a space-time spectral method for the considered inverse problem.

The remainder of this paper comprises five sections. In Section 2, we first describe the inverse problem, and give some preliminary results on the initial value problem associated