Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2013.1125nm

Vol. **6**, No. 2, pp. 424-446
May 2013

A Spectral Method for Neutral Volterra Integro-Differential Equation with Weakly Singular Kernel

Yunxia Wei[1] and Yanping Chen[2,*]

[1] *College of Mathematic and Information Science, Shandong Institute of Business and Technology, Yantai 264005, Shandong, China.*
[2] *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, Guangdong, China.*

Received 23 July 2011; Accepted (in revised version) 27 February 2012

Available online 12 April 2013

Abstract. This paper is concerned with obtaining an approximate solution and an approximate derivative of the solution for neutral Volterra integro-differential equation with a weakly singular kernel. The solution of this equation, even for analytic data, is not smooth on the entire interval of integration. The Jacobi collocation discretization is proposed for the given equation. A rigorous analysis of error bound is also provided which theoretically justifies that both the error of approximate solution and the error of approximate derivative of the solution decay exponentially in L^∞ norm and weighted L^2 norm. Numerical results are presented to demonstrate the effectiveness of the spectral method.

AMS subject classifications: 65R20, 45J05, 65N12

Key words: Neutral Volterra integro-differential equation, weakly singular kernel, Jacobi collocation discretization, convergence analysis.

1. Introduction

We study the neutral Volterra integro-differential equation (VIDE) of the form

$$
\begin{aligned}
y'(t) = a(t)y(t) + b(t) + \int_0^t (t-s)^{-\frac{1}{2}}[K_0(t,s)y(s) \\
+ K_1(t,s)y'(s)]ds, \quad t \in [0,T], \qquad (1.1a)\\
y(0) = y_0, \qquad (1.1b)
\end{aligned}
$$

by the Jacobi spectral collocation method. Here, $a, b : [0,T] \to R$ and $K_0, K_1 : D \to R$ (where $D := \{(t,s) : 0 \le s \le t \le T\}$) are given smooth functions (see [5]). As for

*Corresponding author. *Email addresses:* `yunxiawei@126.com` (Y. Wei), `yanpingchen@scnu.edu.cn` (Y. Chen)

the numerical treatment of VIDE, the reader is referred [2–4, 26, 32, 36] (VIDE with a regular kernel) and [6, 7, 12, 23, 28, 30] (VIDE with a weakly singular kernel), most of these references contain information about other relevant papers.

There are many existing numerical methods for solving VIDE, such as polynomial collocation method [3,7,23,26,27,33], Taylor series method [13], block-by-block method [18, 19], multistep method [21,35] and Runge-Kutta method [2,36]. However, very few works touched the spectral approximation to VIDE. Spectral method has excellent error properties with the so-called "exponential convergence" being the fastest possible. The literature [31] is the first paper proposed a spectral method for Volterra integral equation with a smooth kernel. Subsequently, Y. Chen and T. Tang developed the spectral method for Volterra integral equation with weakly singular kernel in [9, 10].

Usually, the numerical analysis for weakly singular VIDE without neutral term (i.e., $K_1(t,s)=0$) can be based on either of two second-kind Volterra integral equations that are equivalent to the original initial-value problem (1.1a)-(1.1b) (see [5,6]). Its first reformulation has the form

$$y(t)=f(t)+\int_0^t H_1(t,s)y(s)ds, \quad t\in[0,T], \tag{1.2}$$

where

$$f(t)=y_0+\int_0^t b(s)ds, \tag{1.3a}$$

$$H_1(t,s)=a(s)+\int_s^t (v-s)^{-\frac{1}{2}}K_0(v,s)dv. \tag{1.3b}$$

Alternatively, we may consider the equivalent Volterra integral equation for $z(t)=y'(t)$, namely,

$$z(t)=g(t)+\int_0^t H_2(t,s)z(s)ds, \quad t\in[0,T], \tag{1.4}$$

with

$$g(t)=b(t)+\Big(a(t)+\int_0^t (t-s)^{-\frac{1}{2}}K_0(t,s)ds\Big)y_0, \tag{1.5a}$$

$$H_2(t,s)=a(t)+\int_s^t (t-v)^{-\frac{1}{2}}K_0(t,v)dv. \tag{1.5b}$$

But (1.2) and (1.4) are not much suitable for the spectral method since the kernels defined in (1.3b) and (1.5b) have new singularities along the lines $s=0$ and $t=0$, respectively, in addition to those for $s=t$ admitted in [34].

In [15], Y. Jiang considers the Legendre spectral collocation method for VIDE with a smooth kernel, which has smooth solution on the entire interval of integration $[0,T]$ if the