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Quadrature Based Optimal Iterative Methods with Applications in High-Precision Computing

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Abstract. We present a simple yet effective and applicable scheme, based on quadrature, for constructing optimal iterative methods. According to the, still unproved, Kung-Traub conjecture an optimal iterative method based on $n+1$ evaluations could achieve a maximum convergence order of 2^n. Through quadrature, we develop optimal iterative methods of orders four and eight. The scheme can further be applied to develop iterative methods of even higher orders. Computational results demonstrate that the developed methods are efficient as compared with many well known methods.

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Key words: Iterative methods, fourth order, eighth order, quadrature, Newton, convergence, nonlinear, optimal.

1. Introduction

Many problems in science and engineering require solving nonlinear equation

$$f(x) = 0, \tag{1.1}$$

see, e.g., [1–13]. One of the best known and probably the most used method for solving the preceding equation is the Newton's method. The classical Newton method (**NM**) is given as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,3,\cdots, \quad \text{and} \quad |f'(x_n)| \neq 0. \tag{1.2}$$

The Newton's method converges quadratically [1–13]. There exists numerous modifications of the Newton's method which improve the convergence rate (see [1–21] and references therein). This work presents a new quadrature based scheme for constructing optimal iterative methods of various convergence orders. According to the Kung-Traub

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conjecture an optimal iterative method based upon $n+1$ evaluations could achieve a convergence order of 2^n. Through the scheme, we construct optimal fourth order and eighth order iterative methods. Fourth order method requests three function evaluations while the eighth order method requests four function evaluations during each iterative step. The next section presents our contribution.

2. Quadrature based scheme for constructing iterative methods

Our motivation is to develop a scheme for constructing optimal iterative methods. To construct higher order method from the Newton's method (1.2), we use the following generalization of the Traub's theorem (see [16, Theorem 2.4] and [20, Theorem 3.1]).

Theorem 2.1. *Let $g_1(x), g_2(x), \cdots, g_s(x)$ be iterative functions with orders $r_1, r_2, \cdots, r_s$, respectively. Then the composite iterative functions*

$$g(x) = g_1(g_2(\cdots(g_z(x))\cdots))$$

define the iterative method of the orders $r_1 r_2 r_3 \cdots r_s$.

From the preceding theorem and the Newton method (1.2), we consider the fourth order modified double Newton method

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(y_n)}. \end{cases} \tag{2.1}$$

Since the convergence order of the double Newton method is four and it requires four evaluations during each step. Therefore, according to the Kung and Traub conjecture, for the double Newton method to be optimal it must require only three function evaluations. By the Newton's theorem the derivative in the second step of the double Newton method can be expressed as

$$f'(y_n) = f'(x_n) + \int_{x_n}^{y_n} f''(t)\,\mathrm{d}t, \tag{2.2}$$

let us approximate the integral in the preceding equation as follows

$$\int_{x_n}^{y_n} f''(t)\,\mathrm{d}t = \omega_1 f(x_n) + \omega_2 f(y_n) + \omega_3 f'(x_n). \tag{2.3}$$

To determine the real constants ω_1, ω_2 and ω_3 in the preceding equation, we consider the equation is valid for the three functions: $f(t) =$ constant, $f(t) = t$ and $t(t) = t^2$. Which yields the equations

$$\begin{cases} \omega_1 + \omega_2 = 0, \\ \omega_1 x_n + \omega_1 y_n + \omega_3 = 0, \\ \omega_1 x_n^2 + \omega_2 y_n^2 + \omega_3 2 x_n = 2(y_n - x_n). \end{cases} \tag{2.4}$$