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A Compact Difference Scheme for an Evolution Equation with a Weakly Singular Kernel

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Abstract. This paper is concerned with a compact difference scheme with the truncation error of order 3/2 for time and order 4 for space to an evolution equation with a weakly singular kernel. The integral term is treated by means of the second order convolution quadrature suggested by Lubich. The stability and convergence are proved by the energy method. A numerical experiment is reported to verify the theoretical predictions.

AMS subject classifications: 45K05, 65M06, 65M12, 65M15

Key words: Evolution equation, weakly singular kernel, compact difference scheme, stability, convergence, numerical experiment.

1. Introduction

We shall consider a compact difference scheme for the numerical solution of an evolution equation [2, 9, 12, 13, 15, 19]

$$u_t(x,t) - \int_0^t \beta(t-s)u_{xx}(x,s)ds = f(x,t), \qquad 0 < x < 1,\ 0 < t \le T, \tag{1.1}$$

where the kernel $\beta(t) = (\pi t)^{-1/2}$ is singular at $t = 0$, with the boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 < t \le T, \tag{1.2}$$

and the initial condition

$$u(x,0) = v(x), \qquad 0 \le x \le 1. \tag{1.3}$$

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Recall that, for $\gamma > 0$, the γ-th integral $I^{(\gamma)}f(t)$ is defined by the Riemann-Liouville operator (see [14]) as

$$I^{(\gamma)}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s)ds, \qquad t > 0.$$

Thus, the integral term can be viewed as the $1/2$-th integral of $u_{xx}(x,\cdot)$, equation (1.1) is intermediate between the diffusion and the wave equation [4, 5], and it can be termed a fractional partial differential equation of $3/2$-order in time. Equations similar to (1.1) can be found in the modelling of wave propagation involving viscoelastic forces, heat conduction in materials with memory and anomalous diffusion processes [1, 4, 5, 14]. They have recently attracted increasing interest in the physical, chemical and engineering literature (see the numerous papers citing [14]).

A number of people have studied the evolution equation. e.g., Chen, Thomee and Wahlbin [1] used backward Euler scheme in time, piecewise linear finite element method in space, the integral term by means of product integration, and gave the regularity and error boundness of the solution. Lopez-Marcos [9] studied a nonlinear partial integro-differential equation, used one order full discrete difference scheme. Mclean, Thomee [12] employed backward Euler, Crank-Nicolson and second order backward difference scheme, Galerkin finite element method for spatial variables and gave the regularity, stability and error estimate of (1.1)-(1.3). Sanz-Serna [15] studied this type of equations, used backward Euler scheme in time and one order convolution to the integral term, drove error boundedness for smooth and nonsmooth initial value. Xu [19] considered backward Euler and Crank-Nicolson scheme, with one and second order convolution quadrature to the integral term respectively, drove long time error boundness with weights. A practical difficulty of time discretization is that all $\mathbf{U}^n (1 \le n \le N)$ need be stored as they all enter the subsequent equations, which need much memory requirement. In order to conquer this problem, Huang [6] put forward an iterative scheme and reduced the memory requirement. Sloan, Thomee [16] proposed more economical schemes by using quadrature rules with higher order truncation errors.

It is well known that the Crank-Nicolson scheme has $O(k^2 + h^2)$ order accuracy and is unconditionally stable for any step-size ratio k/h^2 for the heat equation. However, it is not an optimum scheme. The six-point implicit difference scheme with minimum truncation error is $(\delta_t U_{j-1}^n + 10\delta_t U_j^n + \delta_t U_{j+1}^n)/12 = (\delta_x^2 U_j^n + \delta_x^2 U_j^{n-1})/2$, whose truncation error is $O(k^2 + h^4)$ [17]. In this paper, we will consider the scheme for the evolution equation with a weakly singular kernel.

Throughout the paper, for $0 \le t \le T$, $0 \le x \le 1$, we assume that there exists a positive constant C such that (see [9, (1.7)])

$$\begin{aligned} &|u_{tt}(x,t)| \le Ct^{-1/2}, \qquad &&|u_{ttt}(x,t)| \le Ct^{-3/2}, \\ &|u_{xxt}(x,0)| \le C, &&|u_{xxtt}(x,t)| \le Ct^{-1/2}. \end{aligned} \tag{1.4}$$

Remark 1.1. For sufficiently smooth $v(x)$ and $f(x,t)$, (1.1)-(1.3) exists a unique solution