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Uniform Convergence of Adaptive Multigrid Methods for Elliptic Problems and Maxwell's Equations

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Abstract. We consider the convergence theory of adaptive multigrid methods for second-order elliptic problems and Maxwell's equations. The multigrid algorithm only performs pointwise Gauss-Seidel relaxations on new degrees of freedom and their "immediate" neighbors. In the context of lowest order conforming finite element approximations, we present a unified proof for the convergence of adaptive multigrid V-cycle algorithms. The theory applies to any hierarchical tetrahedral meshes with uniformly bounded shape-regularity measures. The convergence rates for both problems are uniform with respect to the number of mesh levels and the number of degrees of freedom. We demonstrate our convergence theory by two numerical experiments.

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1. Introduction

In this paper, we study the uniform convergence theory of the adaptive multigrid method for two model problems

$$-\Delta u + u = f \quad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \quad \text{on } \Gamma, \tag{1.2}$$

and

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} + \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \tag{1.3}$$
$$\mathbf{u} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{1.4}$$

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where $\Omega\subset\mathbb{R}^3$ is a Lipschitz polyhedron with boundary $\Gamma=\partial\Omega$, $\boldsymbol{n}$ is the unit outer normal of Γ, and $f\in L^2(\Omega)$, $\mathbf{f}\in(L^2(\Omega))^3$. Problem (1.1)–(1.2) and (1.3)–(1.4) are key model problems for the study of numerical methods for second-order elliptic boundary value problems and quasi-magnetostatic boundary value problems, respectively.

Linear $H_0^1(\Omega)$-conforming finite elements and lowest-order $\boldsymbol{H}_0(\mathbf{curl},\Omega)$-conforming edge elements provide natural finite element trial spaces for the Galerkin discretizations of (1.1)–(1.2) and (1.3)–(1.4), respectively. Here we study optimal iterative solvers for the resulting discrete problems. We remark that optimal approximation entails the use of adaptive finite element methods based on a posteriori error estimates, see [6, 10, 29, 31] for $H^1(\Omega)$-elliptic problems and [5, 11, 26, 39] for $\boldsymbol{H}(\mathbf{curl},\Omega)$-elliptic problems. In this case we can expect the optimal asymptotic convergence rate

$$\left\|u-u_h\right\|_{H^1(\Omega)}\le CN_h^{-1/3},\quad \left\|\mathbf{u}-\mathbf{u}_h\right\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\le CN_h^{-1/3}, \tag{1.5}$$

on families of finite element meshes arising from adaptive refinement. Here, u_h and $\mathbf{u}_h$ are the finite element solutions approximating u and $\mathbf{u}$ respectively, and N_h is the number of elements. An optimal solver delivers a satisfactory approximation of the discrete solution with a number of operations proportional to N_h. In finite element settings, this objective is usually achieved by using geometric multigrid methods, whose convergence theory and optimality on family of uniformly refined meshes have been well established for both $H^1(\Omega)$-elliptic problems [33, 34, 36, 37] and $\boldsymbol{H}(\mathbf{curl},\Omega)$-elliptic problems [1, 15, 17].

To keep the optimal computational cost on locally refined meshes, one must adopt the local multigrid policy [3, 22, 32], which confines relaxations to degrees of freedom on new elements of each mesh level. Clearly this policy makes the computational cost of the local multigrid method proportional to the number of all elements appearing in the local refinement process, and thus proportional to the number of degrees of freedom on the finest mesh. The local multigrid policy with hybrid relaxations for Maxwell's equations are studied in [4, 11, 19, 28]. They show that the local multgird method is very efficient and robust for low-frequency problems on various non-convex domains, and is a good preconditioner for time-harmonic Maxwell's equations [11].

Suppose one seeks for the discrete solution u_h of (1.1)–(1.2) or (1.3)–(1.4) in finite dimensional Hilbert space V_h. For a given partition $\mathcal{T}_h$ of Ω, V_h is usually taken as the finite element space defined over $\mathcal{T}_h$. The multigrid method for solving u_h is designed upon some multilevel decomposition of V_h over a sequence of conforming meshes

$$\mathcal{T}_0\prec\mathcal{T}_1\prec\cdots\prec\mathcal{T}_L:=\mathcal{T}_h.$$

Here $\mathcal{T}_0$ is a quasi-uniform mesh with small number of elements and "$\mathcal{T}_{l-1}\prec\mathcal{T}_l$" means that $\mathcal{T}_l$ is obtained by refining some or all elements in $\mathcal{T}_{l-1}$. The sequence of meshes $\{\mathcal{T}_l\}_{l=0}^L$ can be constructed either by adaptive refinement strategies starting from the initial mesh $\mathcal{T}_0$ (see, e.g., [11,32]), or by some coarsening strategies starting from the final mesh $\mathcal{T}_L$ (see, e.g., [19,35]). Recently, Xu, Chen, and Nochetto [35] present a unified framework for the uniform convergence of multilevel methods for $H^1(\Omega)$–, $\boldsymbol{H}(\mathbf{curl},\Omega)$–, $\boldsymbol{H}(\mathrm{div},\Omega)$–elliptic problems. In [19], Hiptmair and Zheng presented the uniform convergence of the