Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2011.m12si05

Vol. **5**, No. 1, pp. 85-98
February 2012

A Projected Algebraic Multigrid Method for Linear Complementarity Problems

Jari Toivanen[1,*] and Cornelis W. Oosterlee[2]

[1] *Institute for Computational and Mathematical Engineering, Durand Building, room 023A, Stanford University, Stanford, CA 94305, USA and Department of Mathematical Information Technology, Agora, FI-40014 University of Jyväskylä, Finland.*

[2] *Centrum voor Wiskunde en Informatica, Modelling, Analysis and Simulation (MAS2), Kruislaan 413, Amsterdam, The Netherlands and Delft University of Technology, Delft Institute of Applied Mathematics, Mekelweg 4, 2628 CD Delft, The Netherlands.*

Received 13 January 2011; Accepted (in revised version) 05 May 2011

Available 21 December 2011

Abstract. We present an algebraic version of an iterative multigrid method for obstacle problems, called projected algebraic multigrid (PAMG) here. We show that classical algebraic multigrid algorithms can easily be extended to deal with this kind of problem. This paves the way for efficient multigrid solution of obstacle problems with partial differential equations arising, for example, in financial engineering.

AMS subject classifications: 65M55, 65F99, 91G20

Key words: Linear complementarity problem, algebraic multigrid, American options, elasto-plastic torsion problem.

1. Introduction

In this paper we show that the algebraic multigrid (AMG) method, as it is commonly used to solve partial differential equations on unstructured grids in a robust and efficient way, can relatively easily be extended to dealing with obstacle problems. These problems are often encountered in engineering practice, ranging from classical engineering applications like elasto-plastic torsion applications to relatively recent applications occurring, for example, in computational finance.

One of the motivations to develop the AMG method for obstacle problems here is to transfer efficient iterative solution methods of black-box type to application fields like financial engineering. Multigrid has been used in that field, but mainly by academic researchers from universities.

*Corresponding author. *Email addresses:* `toivanen@stanford.edu` (J. Toivanen), `c.w.oosterlee@cwi.nl` (C.W. Oosterlee)

AMG was popularised by the overview article of Ruge and Stüben from 1987 [25], with the basic principles of AMG for so-called M-matrices. Moreover, the software related to the AMG solver described in that article was initially provided as open source software. AMG solvers from the eighties were particularly efficient for matrix problems originating from two-dimensional discrete partial differential equations. At that time, with limited computer resources, this was sufficient. In the early nineties of the previous century, however, computer capacity had increased and partial differential equation software for three-dimensional applications had been developed and a need to solve three-dimensional problems efficiently by black-box methods arose. The AMG solver from the eighties had to be upgraded in terms of reducing its coarse grid operator complexity for three-dimensional problems. A revival of AMG started at that time, for example, by [20,26], as well as in [6]. The resulting AMG solvers with reduced operator complexities due to so-called aggressive coarsening [26], or by other means [8] were parallelized as well for enhanced efficiency, for example in [10, 21], amongst several others. The development of AMG has been described in textbooks such as [4] or [27].

Independent of this development, geometric multigrid, which is explicitly based on grids and structures, has been applied to obstacle problems with partial differential equations (PDEs), also in the eighties of last century. Obstacle problems can be formulated as linear complementarity problems (LCPs), which have a long tradition regarding their efficient numerical solution, see, for example, [7,9]. The LCP formulation is beneficial for iterative solution, since the unknown boundary (as obstacle problems are governed by unknown *free* boundaries) does not appear explicitly and can be obtained in a post-processing step.

In the pioneering paper [3] from 1983, regarding the use of multigrid to this type of problem, Brandt and Cryer introduced the projected full approximation scheme (PFAS) multigrid method for LCPs. Unlike in the basic geometric multigrid correction scheme, which is based on transferring corrections to the numerical solution from coarse to fine grids, each level of PFAS approximates the complete solution of the fine LCP. It is thus based on the non-linear full approximation scheme (FAS) developed by Brandt in [2]. In 1987, Hoppe developed a solver for obstacle problems which employs a multigrid method to solve reduced linear algebraic systems in [13]. A later multigrid approach for obstacle problems was called the monotone multigrid method, developed by Kornhuber in [19].

Reisinger and Wittum proposed in [24] a projected multigrid method for LCPs. It resembles the standard multigrid method more closely than PFAS. The coarse grid right-hand side is the restriction of the defect (residual). In that strain of literature we here present the so-called projected algebraic multigrid (PAMG) method. We will base our algorithm on this latter idea, and use the original Ruge-Stüben AMG framework.

A wide range of financial engineering applications in which obstacle problems occur in the form of LCPs is presented in [14,18]. The basic obstacle problem considered in computational finance is the calculation of the value of an American-style option in a stochastic volatility setting. It leads to the solution of a two-dimensional (plus time) convection-diffusion type PDE with a free boundary. In [28], it has been shown that for the American-style options the theory of linear complementarity applies, so that it is possible to rewrite