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Numerical Analysis of a System of Singularly Perturbed Convection-Diffusion Equations Related to Optimal Control

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Abstract. We consider an optimal control problem with an 1D singularly perturbed differential state equation. For solving such problems one uses the enhanced system of the state equation and its adjoint form. Thus, we obtain a system of two convection-diffusion equations. Using linear finite elements on adapted grids we treat the effects of two layers arising at different boundaries of the domain. We proof uniform error estimates for this method on meshes of Shishkin type. We present numerical results supporting our analysis.

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1. Introduction

Let us consider the following optimal control problem governed by a linear convection-diffusion equation

$$\min_{y,q} J(y,q) := \min_{y,q} \left(\frac{1}{2}\|y - y_0\|_0^2 + \frac{\lambda}{2}\|q\|_0^2 \right) \tag{1.1}$$

subject to

$$Ly := -\varepsilon y'' + ay' + by = f + q, \qquad \text{in } (0,1), \tag{1.2a}$$

$$y(0) = y(1) = 0. \tag{1.2b}$$

We assume

$$0 < \varepsilon \ll 1, \lambda > 0, \quad |a(x)| \geq \alpha > 0, \qquad \text{for } x \in (0,1)$$

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and a, b, f, y_0 to be sufficiently smooth. It is well-known (cf. [10]) that then there is an adjoint state p such that

$$\lambda q + p = 0, \tag{1.3a}$$
$$L^* p = -\varepsilon p'' - a p' + (b - a')p = y - y_0, \tag{1.3b}$$
$$p(0) = p(1) = 0. \tag{1.3c}$$

Consequently, y and p solve the system

$$-\varepsilon y'' + a y' + b y + \frac{1}{\lambda} p = f, \qquad y(0) = y(1) = 0, \tag{1.4a}$$
$$-\varepsilon p'' - a p' + (b - a')p - y = -y_0, \qquad p(0) = p(1) = 0. \tag{1.4b}$$

Discretization methods for system (1.4) (even in the two-dimensional case) are analyzed in [1, 4, 8]. Let us denote the numerical solutions of the given system by (y^N, q^N). Furthermore we denote the numerical solution of the state equation with a given right-hand side by $\tilde{y}^N$ (likewise the numerical solution of the adjoint equation for a given right-hand side by $\tilde{p}^N$). Based on the inequality

$$\|y - y^N\|_0^2 + \lambda \|q - q^N\|_0^2 \le \frac{1}{\lambda} \|p - \tilde{p}^N\|_0^2 + \|y - \tilde{y}^N\|_0^2,$$

the authors are able to estimate first the L_2-norm of $y - y^N$ and $q - q^N$ based on the L_2 errors for the discretization of the primal problem (1.2) and the adjoint problem (1.3) for a given right-hand side. In a second step stability estimates are used to prove error estimates in a stronger norm.

The estimates obtained in these papers contain H^2-norms of y and p which tend, in general, to infinity for $\varepsilon \to 0$. The influence of boundary layer terms is not discussed. But if layers exist, the technique just described is not adequate: in the singularly perturbed case one first estimates in a natural energy norm (and not in the L_2-norm) because optimal L_2 error estimates are more difficult to obtain.

In this paper we present a new technique for analyzing finite element discretizations of problem (1.1) on layer-adapted meshes based on information concerning the layer structure.

System (1.4) is a special case of the following system

$$L(u_1, u_2) := \begin{pmatrix} -\varepsilon u_1'' + a_1 u_1' + b_{11} u_1 + b_{12} u_2 \\ -\varepsilon u_2'' - a_2 u_2' - b_{21} u_1 + b_{22} u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad \begin{matrix} u_1(0) = u_1(1) = 0, \\ u_2(0) = u_2(1) = 0. \end{matrix} \tag{1.5}$$

Assuming

$$a_1, a_2 \ge \alpha > 0, \tag{1.6a}$$
$$b_{11}, b_{22} \ge 0, \tag{1.6b}$$
$$b_{12} b_{21} > 0, \quad |b_{12}|, |b_{21}| \ge \beta > 0, \tag{1.6c}$$