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Simulations of Shallow Water Equations by Finite Difference WENO Schemes with Multilevel Time Discretization

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Abstract. In this paper we study a class of multilevel high order time discretization procedures for the finite difference weighted essential non-oscillatory (WENO) schemes to solve the one-dimensional and two-dimensional shallow water equations with source terms. Multilevel time discretization methods can make full use of computed information by WENO spatial discretization and save CPU cost by holding the former computational values. Extensive simulations are performed, which indicate that, the finite difference WENO schemes with multilevel time discretization can achieve higher accuracy, and are more cost effective than WENO scheme with Runge-Kutta time discretization, while still maintaining nonoscillatory properties.

AMS subject classifications: 65M10, 78A48

Key words: Multilevel time discretization, weighted essentially non-oscillatory schemes, shallow water equations, Runge-Kutta method, high order accuracy.

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1. Introduction

In this paper, the finite difference weighted essential non-oscillatory (WENO) schemes with linear multilevel time discretizations are used to simulate discontinuous flows of the shallow water equations with source terms. The numerical solutions are compared with those of WENO schemes with Runge-Kutta time discretizations. We will be mainly addressing on cost CPU time and resolution.

WENO schemes were firstly developed from essential non-oscillatory (ENO) scheme, which share many advantages with and usually perform better than TVD or TVB schemes,

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because they use an adaptive stencil trying to obtain information from the smoothest regions. ENO schemes started with the classic paper of Harten, *et al*, in 1987 [8,9]. WENO schemes use a convex combination of all candidate stencils instead of the one used as in the original ENO. The first WENO scheme was constructed in 1994 by Liu et al. [13] for the third order finite volume version in one space dimension. Then the third and the fifth order finite difference WENO schemes in multi-space dimensions were constructed. With a general framework for the design of smoothness indicators and nonlinear weights [11], higher order WENO finite difference schemes up to the eleventh order were designed in [3], very high order WENO schemes up to the seventeenth order were developed in [7], and WENO schemes were generalized to triangle meshes [10]. WENO improves upon ENO in robustness, better smoothness of fluxes, better steady convergence, better provable convergence properties, and more efficiency [17].

WENO is a procedure of spatial discretization; namely, it is a procedure to approximate the spatial derivative terms. It forms a very important class of high accuracy numerical methods [11,13], leading to a class of high order finite difference or finite volume methods for hyperbolic conservation laws [2,20]. They give sharp, non-oscillatory discontinuity transitions and at the same time provide high order accurate resolutions for the smooth part of the solution [4,21].

For time-dependent problems, we need accuracy of time discretization as well. There are mainly two different approaches to approximate the time derivative [14,18]. One way is via the classical Lax-Wendroff procedure, which relies on converting all the time derivatives in a temporal Taylor expansion into spatial derivatives, then discretizing the spatial derivatives, so it is also called the Taylor type. The approach can produce the same high order accuracy with a smaller effective stencil. The other way is to use a high order ODE solver, such as Runge-Kutta method or multilevel type. The approach has the advantages of simplicity in concept and in coding and they are easily generalizable to multidimensional problems. Runge-Kutta method is commonly used, but the method costs more CPU time, as for every time step it must iterate k times for the k-th order Runge-Kutta method. Multilevel time discretization methods can make full use of given information with spatial discretization. Stencils of multilevel time discretization are more compact than that of Runge-Kutta methods, since we communicate only with immediate neighbors of cell i to computed u_i^{n+1} from u_i^n, and the CFL number of multilevel method is smaller than that of Runge-Kutta method, while they need the same memory of the cells. Linear multilevel methods are the most commonly used in multilevel procedures.

In this paper, the fifth order WENO finite difference schemes with multilevel type time discretization are used to solve the one-dimensional and two-dimensional shallow water equations. We follow the ideas of Jiang and Shu about the WENO schemes [11], Rogers *et al.* [16] and Xing and Shu [19] about the balance of the flux and the source terms of shallow water equations, and linear multilevel time discretization. This paper is organized as follows. In Section 2, we describe the discretization and numerical method of the shallow water equations. Numerical examples are given to demonstrate the advantages of maintaining high order accuracy, the resolution and cost effective of the constructed schemes in Section 3. Concluding remarks are included in Section 4.