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A Posteriori Error Estimates of Mixed Methods for Quadratic Optimal Control Problems Governed by Parabolic Equations

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Abstract. In this paper, we discuss the a posteriori error estimates of the mixed finite element method for quadratic optimal control problems governed by linear parabolic equations. The state and the co-state are discretized by the high order Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We derive a posteriori error estimates for both the state and the control approximation. Such estimates, which are apparently not available in the literature, are an important step towards developing reliable adaptive mixed finite element approximation schemes for the control problem.

AMS subject classifications: 49J20, 65N30

Key words: A posteriori error estimates, quadratic optimal control problems, parabolic equations, mixed finite element methods.

1. Introduction

Optimal control problems [18] have been extensively utilized in many aspects of the modern life such as social, economic, scientific and engineering applications. Some of the problems require to be solved with efficient numerical methods. Among these numerical methods, finite element method is one of the most useful choices. There have been extensive studies in convergence of finite element approximation for optimal control problems, see, e.g., [1,14,17,25,30]. A systematic introduction of finite element method for optimal control problems can be found for example in [15,26,28].

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In recent years, the adaptive finite element method has been investigated extensively. It has become one of the most popular methods in the scientific computation and numerical modeling. Adaptive finite element approximation ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate but can be characterized by a posteriori error estimators. Hence it is an important approach to boost the accuracy and efficiency of finite element discretizations. There are lots of works concentrating on the adaptivity of various optimal control problems, see, e.g., [3,4,13,16, 19–24]. Note that all the above works aimed at standard finite element method.

In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods, see, e.g., [5]. When the objective functional contains the gradient of the state variable, mixed finite element methods can be used for discretization of the state equation with which both the scalar variable and its flux variable can be approximated in the same accuracy. Recently, in [6–8] we have carried out some primary works on a priori, superconvergence and a posteriori error estimates error estimates for linear elliptic optimal control problems by mixed finite element methods.

In [9], we have derived a posteriori error estimates for parabolic optimal control problems by the lowest order Raviart-Thomas mixed finite element methods. This paper is motivated by the idea of [23]. We shall use the order $k \geq 1$ Raviart-Thomas mixed finite element to discretize the state and the co-state. Due to the limited regularity of the optimal control u in general, we therefore only consider the piecewise constant space. Then we derive a posteriori error estimates for the mixed finite element approximation of the optimal control problem. The estimators for the control, the state and the co-state variables are derived in the sense of $L^\infty(J;L^2(\Omega))$-norm or $L^2(J;L^2(\Omega))$-norm, which are different from the ones in [9]. The optimal control problem that we are interested in is as follows:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T\left(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\}, \tag{1.1}$$

$$y_t(x,t)+\operatorname{div}\boldsymbol{p}(x,t)=f(x,t)+Bu(x,t),\ x\in\Omega,\ t\in J, \tag{1.2}$$

$$\boldsymbol{p}(x,t)=-A(x)\nabla y(x,t),\ x\in\Omega,\ t\in J, \tag{1.3}$$

$$y(x,t)=0,\ x\in\partial\Omega,\ t\in J,\ y(x,0)=y_0(x),\ x\in\Omega, \tag{1.4}$$

where the bounded open set $\Omega\subset\mathbf{R}^2$ is a convex polygon with the boundary $\partial\Omega$, $J=[0,T]$. Let K be a closed convex set in the control space $U=L^2(J;L^2(\Omega))$, $\boldsymbol{p},\boldsymbol{p}_d\in(L^2(J;H^1(\Omega)))^2$, $u,y,y_d\in L^2(J;H^1(\Omega))$, $f\in L^2(J;L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$, B is a bounded linear operator. We assume that the coefficient matrix $A(x)=(a_{ij}(x))_{2\times 2}\in C^\infty(\bar{\Omega};\mathbf{R}^{2\times 2})$ is a symmetric 2×2-matrix and there are constants $c_1,c_2>0$ satisfying for any vector $\mathbf{X}\in\mathbf{R}^2$, $c_1\|\mathbf{X}\|_{\mathbf{R}^2}^2\leq\mathbf{X}^tA\mathbf{X}\leq c_2\|\mathbf{X}\|_{\mathbf{R}^2}^2$.

We assume that the constraint on the control is an obstacle such that

$$K=\left\{u\in U:\ u(x,t)\geq 0,\ \text{a.e. in }\Omega\times J\right\}.$$