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Higher Order Triangular Mixed Finite Element Methods for Semilinear Quadratic Optimal Control Problems

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Abstract. In this paper, we investigate a priori error estimates for the quadratic optimal control problems governed by semilinear elliptic partial differential equations using higher order triangular mixed finite element methods. The state and the co-state are approximated by the order k Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise polynomials of order k ($k \geq 0$). A priori error estimates for the mixed finite element approximation of semilinear control problems are obtained. Finally, we present some numerical examples which confirm our theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: a priori error estimates, semilinear optimal control problems, higher order triangular elements, mixed finite element methods.

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1. Introduction

Optimal control problems governed by semilinear elliptic partial differential equations have been so widely met in all kinds engineering problems. Efficient numerical methods are critical for successful applications of optimal control problems in such cases. Recently, the finite element method of optimal control problems plays an important role in numerical methods for these problems, and the relevant literature is extensive, see, for example, [17, 22, 24, 27].

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Many contributions have been done to the priori error estimates of the standard finite element approximation, see, for example, Falk [11], Geveci [12]. But it is more difficult to obtain such error estimates for optimal control problems where the state equations are nonlinear or where there are inequality state constraints. While a priori error analysis for finite element discretization of optimal control problems governed by elliptic equations is discussed in many publications, see, for example, [16], there are only few published results on this topic for nonlinear optimal control problems, see, for example, Arada and Casas [1], Gunzburger and Hou [15].

In many control problems, the objective functional contains gradient of the state variables. Thus accuracy of gradient is important in numerical approximation of the state equations. In the finite element community, mixed finite element methods should be used for discretization of the state equations in such cases. In computational optimal control problems, mixed finite element methods are not widely used in engineering simulations. In particular there doesn't seem to exist much work on theoretical analysis of mixed finite element approximation of optimal control problems in the literature. More recently, we have done some preliminary work on sharp a posteriori error estimates, error estimates and superconvergence of mixed finite element methods for optimal control problems, see, for example, Chen *et al.* [6–9,23]. However, it doesn't seem to be straightforward to extend these existing techniques to the nonlinear optimal control problems.

For $1 \leq p < \infty$ and m any nonnegative integer let $W^{m,p}(\Omega) = \{v \in L^p(\Omega);\ D^\beta v \in L^p(\Omega) \text{ if } |\beta| \leq m\}$ denote the Sobolev spaces endowed with the norm

$$\| v\|_{m,p}^p = \sum_{|\beta|\leq m} \|D^\beta v\|_{L^p(\Omega)}^p,$$

and the semi-norm $| v |_{m,p}^p = \sum_{|\beta|=m} \|D^\beta v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v\,|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C or c denotes a general positive constant independent of h.

In this paper we derive a priori error estimates of optimal order with respect to all discretization parameters for general semilinear convex quadratic optimal control problems using higher order triangular mixed finite element methods.

We consider the following semilinear quadratic optimal control problems:

$$\min_{u\in K\subset U} \left\{ \frac{1}{2}\|\vec{p} - \vec{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\alpha}{2}\|u\|^2 \right\} \tag{1.1}$$

subject to the state equation

$$\begin{aligned} \mathrm{div}\vec{p} + \phi(y) &= f + Bu, & x &\in \Omega, & (1.2)\\ \vec{p} &= -A\nabla y, & x &\in \Omega, & (1.3)\\ y &= 0, & x &\in \partial\Omega, & (1.4) \end{aligned}$$

where the bounded open set $\Omega \subset \mathbb{R}^2$, is a convex polygon or has the smooth boundary $\partial\Omega$. We shall assume that $f \in H^1(\Omega)$ and $\alpha > 0$ are given, and B is a continuous linear