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Chebyshev Spectral Methods and the Lane-Emden Problem

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Abstract. The three-dimensional spherical polytropic Lane-Emden problem is $y_{rr} + (2/r)y_r + y^m = 0$, $y(0) = 1, y_r(0) = 0$ where $m \in [0,5]$ is a constant parameter. The domain is $r \in [0, \xi]$ where ξ is the first root of $y(r)$. We recast this as a nonlinear eigenproblem, with three boundary conditions and ξ as the eigenvalue allowing imposition of the extra boundary condition, by making the change of coordinate $x \equiv r/\xi$: $y_{xx} + (2/x)y_x + \xi^2 y^m = 0$, $y(0) = 1, y_x(0) = 0, y(1) = 0$. We find that a Newton-Kantorovich iteration always converges from an m-independent starting point $y^{(0)}(x) = \cos([\pi/2]x)$, $\xi^{(0)} = 3$. We apply a Chebyshev pseudospectral method to discretize x. The Lane-Emden equation has branch point singularities at the endpoint $x = 1$ whenever m is not an integer; we show that the Chebyshev coefficients are $a_n \sim \text{constant}/n^{2m+5}$ as $n \to \infty$. However, a Chebyshev truncation of $N = 100$ always gives at least ten decimal places of accuracy — much more accuracy when m is an integer. The numerical algorithm is so simple that the complete code (in Maple) is given as a one page table.

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Key words: Lane-Emden, Chebyshev polynomial, pseudospectral.

1. Introduction

The Lane-Emden problem is

$$y_{rr} + (2/r)y_r + y^m = 0, \qquad y(0) = 1,\ y_r(0) = 0, \tag{1.1}$$

where $m \in [0,5]$ is a constant parameter. (This is the three-dimensional spherical polytropic case whose astrophysical context is given in the book by the Nobel Laureate "Black

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Hole" Chandrasekhar [7]; other variants are described in [10].) The goal of the astrophysical problem is to integrate this equation from the origin to its first zero, $r = \xi$. It is helpful to rescale the problem by defining

$$r = \xi x, \qquad x \equiv r/\xi. \tag{1.2}$$

The problem becomes

$$y_{xx} + (2/x)y_x + \xi^2 y^m = 0, \qquad y(0) = 1, y_x(0) = 0, y(1) = 0. \tag{1.3}$$

This is a nonlinear eigenvalue problem with the location of the first zero ξ as the eigenvalue. The eigenvalue is chosen so that the extra boundary condition is satisfied. Analytical solutions are known only for the exponent m equal to 0, 1 or 5 as catalogued in Table 1.

Table 1: Analytical exact solutions.

m	$y(r;m)$	ξ [first zero of $y(r;m)$]
0	$1-(1/6)r^2$	$\sqrt{6}$
1	$\sin(r)/r$	π
5	$1/\sqrt{1+r^2/3}$	∞

The Lane-Emden problem has a long history. Numerical tables for selected values of m were published as early as 1932. A small subset of the available literature is given in the bibliography table. This problem has become one of those benchmarks which are revisited repeatedly as every new numerical method is tested against it. In spite of this vast literature, recent applications of higher spectral methods have sloughed over important difficulties.

First, the Lane-Emden equation is *singular* at the right endpoint (where $y(1) = 0$) whenever the order m is not equal to an integer. Singularities degrade the usual exponential rate of convergence of a spectral method to a finite order rate of convergence. That is to say, the error falls proportional to $1/N^k$ for some constant k, the so-called "algebraic order of convergence", where k depends on the type of singularity as will be explained in more detail below. Fortunately, it is possible to modify spectral methods so as to recover an exponential rate of convergence as will be explained later. Second, the Lane-Emden problem is a nonlinear eigenvalue problem. Although the differential equation is the second order, we need to satisfy three boundary conditions. This is possible because the problem also contains an eigenparameter space ξ which must be determined simultaneously with the solution to the differential equation.

2. Numerical strategies

One strategy is based upon the following theorem.

Theorem 2.1. *Suppose that $w(x,\xi)$ solves $w_{xx}+(2/x)w_x+\xi^2 w = 0$ with $w(0) = 1, w_x(0) = 0, w(1) = 0$. Then $v \equiv \omega w(x,\xi)$ solves $v_{xx} + (2/x)v_x + \Xi^2 v^m = 0$ with $v(0) = \omega, v_x(0) =$*