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Spectral Matrix Conditioning in Cylindrical and Spherical Elliptic Equations

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Abstract. In the spectral solution of 3-D Poisson equations in cylindrical and spherical coordinates including the axis or the center, it is convenient to employ radial basis functions that depend on the Fourier wavenumber or on the latitudinal mode. This idea has been adopted by Matsushima and Marcus and by Verkley for planar problems and pursued by the present authors for spherical ones. For the Dirichlet boundary value problem in both geometries, original bases have been introduced built upon Jacobi polynomials which lead to a purely diagonal representation of the radial second-order differential operator of all spectral modes. This note details the origin of such a diagonalization which extends to cylindrical and spherical regions the properties of the Legendre basis introduced by Jie Shen for Cartesian domains. Closed form expressions are derived for the diagonal elements of the stiffness matrices as well as for the elements of the tridiagonal mass matrices occurring in evolutionary problems. Furthermore, the bound on the condition number of the spectral matrices associated with the Helmholtz equation are determined, proving in a rigorous way one of the main advantages of the proposed radial bases.

AMS subject classifications: 65M10, 78A48

Key words: Poisson equation, cylindrical and spherical coordinates, Fourier expansion, spherical harmonics, Jacobi polynomials, spectral methods, condition number.

1. Introduction

The spectral method is the par excellence approach for solving elliptic problems in geometrically simple domains. For instance, for the simplest plane domain—a rectangle—the solution can be approximated by a double expansion of product functions defined conveniently so that the 2-D Laplace operator is reduced to a pair of ordinary differential equations, see, *e.g.*, Gustafson [6, p.144]. In this way, variable separation is recognized to be the leading lady of the play of the solution procedure, whenever the partial differential equation is homogeneous.

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For nonhomogeneous equations, variable separation changes its character substantially on the stage of the numerical dance but it remains starring. The expansion functions, that in the homogeneous case are the product of analytical solutions to ODEs, are now replaced by basis functions that can be chosen more freely. For instance, always with reference to the Poisson equation in a rectangle, polynomials in the Cartesian coordinate variables can be employed. This leads to a matrix representation of the terms of the elliptic equation usually written by the direct product notation. Alternatively, the terms of the constant coefficient discrete equation can be read very conveniently as the pre- and post-multiplication of the rectangular array of unknown coefficients by the operator matrices associated with the two spatial directions. Such an interpretation lends itself to direct solution algorithms which are based on the diagonalization of the pre- and/or post-multiplying matrices, as it was proposed in the classical paper of Haidvogel and Zang [7]. The diagonalization procedure acts independently on each spatial direction and represents therefore a numerical counterpart of the analytical method of separation of variables, as pointed out by Boyd [5, p. 314].

On the other hand, the relative arbitrariness in the selection of the spectral basis functions for nonhomogeneous problems allows one to search for appropriate bases that give the most convenient matrices from the viewpoint of sparsity and conditioning. For the Cartesian Laplacian, Jie Shen has introduced a basis of Legendre polynomials [13] which is optimal for the solution of Dirichlet problems for second-order elliptic equations. In fact, Shen's functions constitute a hierarchical basis and lead to the simplest representation of the second derivative operator: the identity matrix. Furthermore, the spectral components of the unknown in a given direction are coupled only very weakly, as revealed by the tridiagonal profile of the mass matrix, when even-odd mode reordering is exploited. The good properties of this basis are also revealed by the condition number of the basic matrix of the Galerkin spectral solver which goes as N^2, where N is the number of the basis functions in one spatial direction, instead of N^4 as in pure Legendre or Chebyshev polynomial approximations. Thus, a double diagonalization to build a direct solver *à la* Haidvogel and Zang can be efficiently employed and provides an optimally stable solution procedure.

But what happens to the diagonalization and the direct solution strategy for fully 3-D problems in a cylindrical or spherical domain which includes the axis or the centre? Almost invariably, the first step consists in a Fourier expansion of the angular dependence around the cylindrical or polar axis. In this way, the 3-D elliptic equation is transformed into a sequence of 2-D elliptic equations for the Fourier expansion coefficients of the unknown. Then, the dependence on the radial variable must be tackled, having in mind that the equation becomes singular for $r \to 0$. The singularity is actually only a mathematical artifact of the coordinate system employed, while the 3-D solution to any physical problem should not suffer any loss of differentiability there. As a consequence, the expansion coefficients of an infinitely differentiable function expressed in cylindrical or spherical coordinates by a direct product basis must satisfy suitable regularity conditions as $r \to 0$.

In the cylindrical case, a double expansion in direct product form can be employed to represent the dependence on the radial and axial variable, see, *e.g.*, [9, 14]. This approach is simple but it presents the drawback of retaining more degrees of freedom than strictly