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Best Rank-One Approximation of Fourth-Order Partially Symmetric Tensors by Neural Network

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Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. Our purpose is to compute the multi-partially symmetric rank-one approximations of higher-order multi-partially symmetric tensors. A special case is the partially symmetric rank-one approximation for the fourth-order partially symmetric tensors, which is related to the biquadratic optimization problem. For the special case, we implement the neural network model by the ordinary differential equations (ODEs), which is a class of continuous-time recurrent neural network. Several properties of states for the network are established. We prove that the solution of the ODE is locally asymptotically stable by establishing an appropriate Lyapunov function under mild conditions. Similarly, we consider how to compute the multi-partially symmetric rank-one approximations of multi-partially symmetric tensors via neural networks. Finally, we define the restricted M-singular values and the corresponding restricted M-singular vectors of higher-order multi-partially symmetric tensors and design to compute them. Numerical results show that the neural network models are efficient.

AMS subject classifications: 15A18, 15A69, 65F15, 65F10

Key words: Biquadratic optimization, multi-partially symmetric tensor, M-eigenvalues and eigenvectors, M-singular values and singular vectors, restricted M-singular values and singular vectors, neural network, locally asymptotic stability, Lyapunov stability theory.

1. Introduction

An increasing number of signal processing, data analysis and higher-order statistics, as well as independent component analysis [2, 6, 8, 9, 30] involve the manipulation of

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quantities with more than two indices. These high order equivalents of vectors (first order) and matrices (second order) are called *higher-order tensors*, *multi-dimensional matrices*, or *multiway arrays*.

A tensor is an N-dimensional array of numbers denoted by $\mathscr{A} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$ with entries

$$a_{i_1 i_2\cdots i_N} \in \mathbb{R} \quad \text{for} \quad i_n = 1,2,\cdots,I_n \quad \text{with} \quad n = 1,2,\cdots,N.$$

The fourth-order partially symmetric tensors have received much attention in recent years [10,17,25,32,38,42,45]. Zhang et al. [46] proved that the best rank-one approximation of a symmetric tensor is its best symmetric rank-one approximation. Similarly, we can prove that the best rank-one approximation of a fourth order partially symmetric tensor is its best partially symmetric rank-one approximation. Hence, our goal is to focus on the following rank-one approximation problem of a partially symmetric tensor $\mathscr{A} \in \mathbb{R}^{I_1\times I_1\times I_2\times I_2}$ by real-valued neural networks: finding two unit vectors $\mathbf{x}_n \in \mathbb{R}^{I_n}$ $(n = 1,2)$ and $\sigma \in \mathbb{R}$ to minimize

$$\sum_{i_1,i_2=1}^{I_1}\sum_{i_3,i_4=1}^{I_2}\left[a_{i_1i_2i_3i_4} - \sigma\cdot(x_{1,i_1}x_{1,i_2}x_{2,i_3}x_{2,i_4})\right]^2. \tag{1.1}$$

The above problem arises from the strong ellipticity condition problem in solid mechanics [20,21,35,41] and the entanglement problem in quantum physics [10,36].

There exists several numerical methods to solve the minimization problem (1.1), such as the alternating least squares (ALS) or the higher-order power method (HOPM) [42] and the semidefinite programming (SDP) relaxations [45].

Multiparameter optimizations (unconstrained and constrained) can be accomplished by implementing the dynamical gradient systems, whose states evolve in time towards steady-state solution critical point solutions [7,24,34]. The essence of a neural network, represented as a dynamic system, is to minimize an nonnegative energy function. The dynamical system is typically described in the form of first-order ODEs. For an given initial value, the ODEs will approach the equilibrium point which corresponds to the solution of the underlying optimization problem. There is an important requirement: the energy function decreases monotonically as the solution of the ODEs approach an equilibrium point. In particular, Che et al. [4] present the neural dynamic networks to compute a local optimal rank-one approximation of a real-valued tensor.

In this paper, we propose a new approach for solving the minimization problem (1.1) by neural networks. The proposed model of neural networks can be described by the first-order ODEs, and considered as a generalization of [12,14]. We prove the locally asymptotic stability of solutions of ODEs by establishing an appropriate Lyapunov function. Similar to the fourth-order partially symmetric tensors, we define the real-valued higher-order multi-partially symmetric tensor. We show that the best multi-partially symmetric rank-one approximation of a real-valued higher-order multi-partially symmetric tensor is its best rank-one approximation. Analogous to the fourth-order partially symmetric tensors, we also consider the rank-one approximation of a real-valued higher-order multi-partially symmetric tensor by neural networks. Any numerical method to solve systems of differen-