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REVIEW ARTICLE

Preconditioners for Incompressible Navier-Stokes Solvers[†]

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Abstract. In this paper we give an overview of the present state of fast solvers for the solution of the incompressible Navier-Stokes equations discretized by the finite element method and linearized by Newton or Picard's method. It is shown that block preconditioners form an excellent approach for the solution, however if the grids are not to fine preconditioning with a Saddle point ILU matrix (SILU) may be an attractive alternative. The applicability of all methods to stabilized elements is investigated. In case of the stand-alone Stokes equations special preconditioners increase the efficiency considerably.

AMS subject classifications: 65F10, 65N30, 76D05

Key words: Navier-Stokes equations, finite element method, block preconditioners, SIMPLE-type schemes, iterative methods, incompressible fluids.

1. Introduction

The numerical solution of the incompressible Navier-Stokes equations has been a challenge for over 50 years. Was the attention in the early days focused on the discretization, nowadays efficient solution is a hot topic.

In this paper we deal with efficient solution of the stationary, laminar incompressible Navier-Stokes equations, discretized by the finite element method. Due to the absence of the pressure in the continuity equation, discretization of these equations requires special care. Finite element approximations can not be chosen at random, but the elements must satisfy the *Ladyshenskaya-Brezzi-Babuška* (LBB) condition in order to guarantee stability of the discretization [1,2]. This condition usually requires an approximation of the velocity

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that is one degree higher than that of the pressure. Most elements satisfying the LBB condition have a quadratic velocity approximation [3].

The alternative is to stabilize the elements by adapting the continuity equation in some way, thus allowing for example equal order interpolation [4]. Although such an approximation makes the implementation more easy, in general the price to be paid is a less accurate pressure.

Solution of the non-linear Navier-Stokes equations requires a suitable linearization technique like Newton or Picard (successive substitution) [3]. The result is a system of linear equations of saddle-point type, containing a zero block on the main diagonal corresponding to the continuity equation. Direct solution of such a system requires a suitable renumbering technique to avoid zero pivots [5]. A possible way of avoiding this problem is to segregate pressure and velocity computation. An approach in this direction is the so-called penalty function approach [3], which has been successfully applied to medium-sized 2d problems. Due to the presence of the penalty parameter, the condition of the linear system is very high and iterative solution is impossible. The alternative is to use Uzawa-type iteration schemes, which unfortunately converge very slowly [6].

During the last decade various iterative solution techniques to solve saddle-point type equations, have been the subject of research. Some methods are focused on clever renumbering schemes in combination with a classical iterative approach, like for example the SILU scheme [5] and ILU schemes proposed by Wille et al. [7–9].

Other methods are based on segregation. The system of equations is split into a velocity and a pressure part. The complete system is solved by an iterative method, but the necessary preconditioner is based on the splitting. We distinguish between block preconditioners and preconditioners based on the classical SIMPLE method of Patankar [10]. A number of block-preconditioners have been devised, for example the Pressure-Convection Diffusion commutator (PCD) of Kay, Logan and Wathen [11, 12], the Least Squares Commutator (LSC) by Elman, Howle, Shadid, Shuttleworth and Tuminaro [13], the Augmented Lagrangian Approach (AL) of Benzi and Olshanskii [14], the Artificial Compressibility (AC) preconditioner [15] and the Grad-Div (GD) preconditioner [15]. For an overview of block preconditioners, we refer to [16–18].

SIMPLE-type preconditioners form a different class of preconditioners, although they can also be considered as a block preconditioner. Besides the standard SIMPLE preconditioner, also improvements like SIMPLER and variants have been developed.

In this paper we present a survey of the most popular solvers for the incompressible Navier-Stokes equations. In the case of the incompressible Stokes equations, we derive some special methods, that outperform the generally applicable solvers. New in this paper is that we investigate the possibility to extend all solvers to stabilized elements. Furthermore we discuss termination criteria and propose a method to improve these criteria. Solvers are compared on the basis of implementation issues, dependence on tuning parameters and numerical experiments.

The remaining part of this paper is as follows. In Section 2 the discretization of the incompressible Navier-Stokes by the Finite element discretization is considered. Section 3 deals with block preconditioners and in Section 4 SIMPLE and its variants are discussed. In