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Bivariate Simplex Spline Quasi-Interpolants

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Abstract. In this paper we use the simplex B-spline representation of polynomials or piecewise polynomials in terms of their polar forms to construct several differential or discrete bivariate quasi interpolants which have an optimal approximation order. This method provides an efficient tool for describing many approximation schemes involving values and (or) derivatives of a given function.

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1. Introduction

The construction of most classical approximants of a given data set or a function usually requires the solving of linear systems. Spline quasi-interpolants are local approximants avoiding this problem, so they are very convenient in practice. In general, a quasi-interpolant of a given function f is obtained as a linear combination of some elements of a suitable set of basis functions. In order to achieve local control, these functions are required to be positive, to ensure the stability and to have small local supports. The coefficients of the linear combination are the values of linear functionals depending on f and (or) its derivatives or integrals.

Various methods for building univariate and bivariate quasi-interpolants have been developed in the literature see for examples [1,5,11,16,22], and references therein. Usually, the appropriate basis functions are B-splines. In the univariate case, these B-splines are well-known and satisfy the above required properties. So the study of the corresponding quasi-interpolants is well developed and they are used for solving several problems in different fields. However, in the bivariate setting, the construction of B-splines which are positive and form a stable basis of spline space is difficult, even impossible except on very special cases, such as the Powell-Sabin basis introduced by P. Dierckx in [13] and the simplex B-spline basis. Several authors have presented collections of simplex B-splines which guarantee such properties. The first collection has been introduced by de

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Boor in [10], which proposes to consider the cylinder or "Slab" $\mathbb{R}^s \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is a suitable convex hull polytope of unit volume, and subdivide this Slab into non trivial simplices. Since then, the above theory has been explored extensively by Dahmen and Micchelli in [7, 8] and by Höllig in [15]. From the point of view of blossoming (i.e., using the polar form), two collections are proposed. The first one, called triangular B-splines (or DMS B-splines), introduced by Dahmen, Micchelli and Seidel in [9]. The second one, studied by Neamtu [20], is the natural generalization of univariate B-splines.

Simplex B-splines are a powerful and flexible class of geometric objects defined over arbitrary, non-rectangular domains. Despite their great potential advantages and their interesting theoretical properties, practical techniques and computational tools with simplex B-splines are less-developed. The first works which use theoretical simplex B-splines in approximation were introduced by W. Dahmen and C. A. Micchelli in [7, 8]. They proposed an approximation process to construct a differential simplex spline quasi-interpolant which generalize the quasi-interpolant introduced by C. de Boor and G. J. Fix [12]. In this paper, we propose a method for constructing discrete and differential simplex spline quasi-interpolants which reproduce the space of polynomials of degree at most n or the whole space of simplex splines associated with a given set of knots. This method (see [23], for instance) is based on the polar form of a chosen local polynomial approximant like local interpolants or other operators having the optimal approximation order. It also requires a collection of simplex B-splines which allows to express any bivariate polynomial p or simplex spline s as a combination of normalized simplex B-splines; and coefficients in the combinations should be given in terms of the polar forms of p or s. This latter condition is only satisfied by the DMS B-splines and Neamtu B-splines. In our work we use DMS B-splines to describe different schemes for differential and discrete quasi-interpolants.

The paper is organized as follows. In Section 2 we give some definitions and properties of bivariate simplex B-splines. In Section 3 we introduce the B-spline representation of all bivariate polynomials or DMS splines over a triangulation Δ of a bounded domain $D \subset \mathbb{R}^2$, in terms of their polar forms. In Section 4 we apply the approach introduced in [23] to DMS B-splines. Then we describe some differential and discrete simplex spline quasi interpolants which reproduce bivariate polynomials and provide the full approximation order in the space of bivariate simplex splines. In Section 5 we give some upper bounds of the infinity norms of some families of discrete quasi-interpolants. Finally, some numerical examples are proposed in Section 6.

2. Bivariate simplex B-splines

For any ordered set of affinely independent points $W = \{w_0, w_1, w_2\} \subset \mathbb{R}^2$ and any point $x \in \mathbb{R}^2$ we define

$$d(W) = det \begin{pmatrix} 1 & 1 & 1 \\ w_0 & w_1 & w_2 \end{pmatrix},$$