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Smoothing Newton-Like Method for the Solution of Nonlinear Systems of Equalities and Inequalities

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Abstract. In this paper, we present a smoothing Newton-like method for solving nonlinear systems of equalities and inequalities. By using the so-called max function, we transfer the inequalities into a system of semismooth equalities. Then a smoothing Newton-like method is proposed for solving the reformulated system, which only needs to solve one system of linear equations and to perform one line search at each iteration. The global and local quadratic convergence are studied under appropriate assumptions. Numerical examples show that the new approach is effective.

AMS subject classifications: 65K05, 90C30

Key words: Nonlinear systems of equalities and inequalities, semismooth function, smoothing Newton method, global convergence, local quadratic convergence.

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1. Introduction

In this paper, we present a smoothing Newton-like method for the numerical solution of nonlinear systems of equalities and inequalities defined by

$$\begin{cases} c_i(x)=0, & i\in E=\{1,2,\cdots,m_e\}, \\ c_i(x)\le 0, & i\in I=\{m_e+1,\cdots,m\}, \end{cases} \tag{1.1}$$

where $c_i(x):\mathbb{R}^n\to\mathbb{R}, i=1,\cdots,m$ are continuously differentiable. Throughout this paper, we assume that the solution set of (1.1) is nonempty.

Systems of nonlinear equalities and inequalities appear in a wide variety of problems in applied mathematics. These systems play a central role in the model formulation design

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and analysis of numerical techniques employed in solving problems arising in optimization, power system, nonlinear complementarity and variational inequalities, etc. Many researchers considered the problem, especially for the numerical methods, see, e.g., [1–8]. Daniel in [4] considered Newton's methods for solving the problem ; Polyak in [5] discussed a gradient method; Burke and Han in [6] presented a Gauss-Newton approach to solve generalized inequalities; Dennis et al. firstly presented trust-region methods for solving (1.1) and proved the global convergence under certain conditions in [7]; Tong-Zhou in [8] also studied another trust-region method for the problem and proved the global convergence under general conditions.

In the last decade, a class of popular numerical methods, namely, the so-called semismooth Newton methods, has been studied extensively for solving semismooth equations, see, e.g., [9–11, 16] and references therein. The typical characteristic of the semismooth Newton is twofold: it extends the classical Newton method for nonsmooth equations; it enjoys the same convergent property such as the locally superlinear convergence. The semismooth Newton method was firstly presented by Qi-Sun in [16]. Then it is studied extensively and used for solving many mathematical problems, such as large scale nonlinear complementarity, variational inequalities and the KKT system of optimization problems, etc (see [10]). Following the semismooth Newton methods, another related numerical method, called the smoothing Newton method, is also presented to help the calculation of generalized derivative of nonsmooth functions, see, e.g., [12–15, 18]. The key idea of smoothing Newton method is to approximate the nonsmooth function $F(x)$ by a smooth function $f(x,\varepsilon)$, where ε is called smoothing parameter. Then the generalized derivative $\partial F(x)$ is approximated by $f'(x,\varepsilon)$ with respect to the variable x. The main advantage of the smoothing Newton methods is that it still retains the nice convergence property. There are two kinds of smoothing methods. One is to handle ε as a parameter, which is updated step by step in iterations (see [12]). The other one is to handle ε as a variable. Then an extended system of equations is set and solved by Newton-type methods. Both smoothing Newton methods are proved to have nice convergence property.

We note that the methods proposed in [7, 8] are based on the least-squares approach, which implies that the system (1.1) is solved by optimization methods. As we all know, compared with the optimization-based methods, solving a system of equations is much easier and it has less calculating cost. Furthermore, the solution of the optimization-based method is the stationary point of the optimization problem, and may not be the solution of (1.1). Therefore, our aim in this paper is to set a system of equations for solving (1.1). To this end, we only consider the case that $m \le n$. We transfer (1.1) into a system of semismooth equations firstly, then a smoothing algorithm is presented for solving the equivalent system. Now we introduce the transformation of (1.1). Denote the maximal function by $\max\{0, c_i(x)\}$. The inequalities

$$c_i(x) \le 0, \quad i \in I$$

can be changed equivalently into the following equations:

$$\max\{0, c_i(x)\} = 0, \quad i \in I.$$