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A Parallel Algorithm for Adaptive Local Refinement of Tetrahedral Meshes Using Bisection

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Abstract. Local mesh refinement is one of the key steps in the implementations of adaptive finite element methods. This paper presents a parallel algorithm for distributed memory parallel computers for adaptive local refinement of tetrahedral meshes using bisection. This algorithm is used in PHG, *Parallel Hierarchical Grid (*`http://lsec.cc.ac.cn/phg/`*)*, a toolbox under active development for parallel adaptive finite element solutions of partial differential equations. The algorithm proposed is characterized by allowing simultaneous refinement of submeshes to arbitrary levels before synchronization between submeshes and without the need of a central coordinator process for managing new vertices. Using the concept of *canonical refinement*, a simple proof of the independence of the resulting mesh on the mesh partitioning is given, which is useful in better understanding the behaviour of the bisectioning refinement procedure.

AMS subject classifications: 65Y05, 65N50

Key words: Adaptive refinement, bisection, tetrahedral mesh, parallel algorithm, MPI.

1. Introduction

The local refinement algorithms of simplicial meshes, mainly meshes composed of triangles in two-dimensions, tetrahedra in three-dimensions, have been extensively studied by many authors. In adaptive finite element computations, two types of refinement algorithms are widely used: the regular refinement and the bisectioning refinement. The regular refinement consists of simultaneously bisecting all edges of the triangle or tetrahedron to be refined, producing 4 smaller triangles or 8 smaller tetrahedra. Since the regular refinement cannot generate locally refined conforming meshes, either special numerical algorithms are designed to handle the hanging nodes, or it is combined with other types of refinement (e.g., the red/green algorithm) to produce a conforming mesh, see,

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e.g., [19–21]. While with the bisectioning refinement, only one edge of the triangle or the tetrahedron, called the *refinement edge*, is bisected, producing 2 smaller triangles or tetrahedra, as illustrated by Fig. 1. The main advantage of bisectioning refinement is that it can naturally produce locally refined conforming meshes and nested finite element spaces. The main problems with bisectioning refinement are how to select the refinement edge such that triangles or tetrahedra produced by successive refinements do not degenerate, and the refinement procedure on a given mesh terminates in a finite number of steps, producing a conforming mesh which is ready for further refinements.

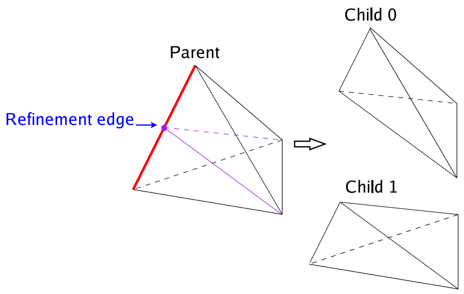

Figure 1: Bisection of a tetrahedron.

The methods for selecting the refinement edge proposed by various authors can be classified into two categories, namely the *longest edge approach* and the *newest vertex approach* (the latter is also called the *newest node approach* by some authors).

The longest edge based algorithms are proposed and mainly studied by Rivara *et al.* [14, 15]. In these algorithms, triangles or tetrahedra are always bisected using one of their longest edges, the finite termination of the refinement procedure is obvious because when traversing from one simplex to another in order to make the resulting mesh conforming, one steps along paths of simplices with longer longest edges. In two dimensions, it can be shown that triangles produced by repeated bisection using the longest edge approach have angles bounded away from 0 and π. But in three dimensions, The non-degeneracy of tetrahedra produced by repeated bisections is still open.

The newest vertex approach was first proposed for two dimensional triangular meshes by Sewell [5], and was generalized to three dimensions by Bänsch [4]. Mitchell did much of the early work on the newest vertex bisection for triangular meshes [6,7]. Maubach [10] further generalized the method for $n+1$-simplicial meshes in n dimensions. More recent work on the newest vertex algorithms is described in the papers of Kossaczky [1], Liu and Joe [3], and Arnold *et al.* [2]. Though the concept of the newest vertex approach is very simple in two dimensions: once the refinement edge for the triangles in the initial mesh is determined, the refinement edge of a newly generated triangle is the edge opposite to its newest vertex, its generalization to three dimensions is highly non-trivial, and the algorithms proposed by various authors are essentially equivalent, but use different interpretations. It is theoretically proved that tetrahedra generated by these algorithms belong to a finite number of similarity cases, which ensures non-degeneracy of tetrahedra produced by repeated bisections.