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Approximation of Derivative for a Singularly Perturbed Second-Order ODE of Robin Type with Discontinuous Convection Coefficient and Source Term

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Abstract. In this paper, a singularly perturbed Robin type boundary value problem for second-order ordinary differential equation with discontinuous convection coefficient and source term is considered. A robust-layer-resolving numerical method is proposed. An ε-uniform global error estimate for the numerical solution and also to the numerical derivative are established. Numerical results are presented, which are in agreement with the theoretical predictions.

AMS subject classifications: 65L10, CR G1.7

Key words: Singular perturbation problem, piecewise uniform mesh, discrete derivative, discontinuous convection coefficient, Robin boundary conditions, discontinuous source term.

1. Introduction

The theory of singular perturbation is not a settled direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving; large or small parameters, become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in Fluid Mechanics, edge layers in Solid Mechanics, skin layers in Electrical Applications and shock layers in Fluid and Solid Mechanics.

Various methods for the numerical solution of problem involving singularly perturbed second-order ordinary differential equations with non-smooth data (discontinuous source

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term/convection coefficient) using special piecewise uniform meshes (Shishkin mesh and Bakhvalov mesh) have been considered widely in the literature (see [11–15] and references therein). While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications. It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution.

Li, Shishkin and Shishkina [5] obtained an approximation of the solution and its derivative for the singularly perturbed Black-Scholes equation with non-smooth initial data. In [2–4, 6], approximations to the normalized derivative $\varepsilon(\partial/\partial x)u(x,t)$, that is, the first order spatial derivative multiplied by the parameter ε, were considered. In [1], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain where as Kopteva and Stynes [8] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. It may be noted that the source term and convection coefficient are smooth for the problem considered in [1,8]. Priyadharshini and Ramanujam [9] estimated the scaled derivative for a singularly perturbed reaction-convection-diffusion problem with two parameters. To the best of our knowledge, it seems no work has been reported in the literature for finding approximation to scaled derivatives of the solution for problems having discontinuous convection coefficient for both upwind and hybrid finite difference schemes on Shishkin mesh.

Motivated by the works given in [10, 12], the present paper considers singularly perturbed second order ordinary differential equation with discontinuous coefficients. Since derivatives are related to flux or drag in physical and chemical applications, we obtain parameter-uniform approximations not only to the solution but also to its derivatives. Thus in this paper, motivated by the work of [8], bounds on the errors in approximating the first derivative of the solution with weight in the fine mesh where as without weight in the coarse mesh are obtained.

Note: Through out this paper, C denotes a generic constant (sometimes subscripted) is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Let $y : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\| y \| = \sup_{x\in D} |y(x)|$.

2. Continuous problem

A singularly perturbed convection-diffusion equation in one dimension with discontinuous convection coefficient and source term is considered on $\Omega = (0,1)$. A single discontinuity is assumed to occur at a point $d \in \Omega$. It is convenient to introduce the notation