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The Jackson Inequality for the Best L^2-Approximation of Functions on $[0,1]$ with the Weight x

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Abstract. Let $L^2([0,1],x)$ be the space of the real valued, measurable, square summable functions on $[0,1]$ with weight x, and let $\mathscr{L}_n$ be the subspace of $L^2([0,1],x)$ defined by a linear combination of $J_0(\mu_k x)$, where J_0 is the Bessel function of order 0 and $\{\mu_k\}$ is the strictly increasing sequence of all positive zeros of J_0. For $f \in L^2([0,1],x)$, let $E(f,\mathscr{L}_n)$ be the error of the best $L^2([0,1],x)$, i.e., approximation of f by elements of $\mathscr{L}_n$. The shift operator of f at point $x \in [0,1]$ with step $t \in [0,1]$ is defined by

$$T(t)f(x) = \frac{1}{\pi}\int_0^\pi f\left(\sqrt{x^2+t^2-2xt\cos\theta}\right)d\theta.$$

The differences $(I-T(t))^{r/2}f = \sum_{j=0}^\infty (-1)^j \binom{r/2}{j} T^j(t)f$ of order $r \in (0,\infty)$ and the $L^2([0,1],x)-$ modulus of continuity $\omega_r(f,\tau) = \sup\{\|(I-T(t))^{r/2}f\| : 0 \le t \le \tau\}$ of order r are defined in the standard way, where $T^0(t) = I$ is the identity operator. In this paper, we establish the sharp Jackson inequality between $E(f,\mathscr{L}_n)$ and $\omega_r(f,\tau)$ for some cases of r and τ. More precisely, we will find the smallest constant $\mathscr{K}_n(\tau,r)$ which depends only on n, r, and τ, such that the inequality $E(f,\mathscr{L}_n) \le \mathscr{K}_n(\tau,r)\omega_r(f,\tau)$ is valid.

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1. Introduction

1.1. Some histories

The Jackson inequalities with the first and higher modulus of continuity in various function spaces of one and several variables have a long history. The *Jackson inequality* usually means the following relation between the value $d(f,L,X)$ of the best approximation of a

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function f in a normed function space X by elements of a subspace L and the structure characterization of the function f in terms of some seminorm (or quasi-seminorm) $|\cdot|_X$:

$$d(f,L,X) \leq K(L,X)|f|_X \quad \text{for all} \quad f \in X. \tag{1.1}$$

The greatest lower bound of the $K(L,X)$ is called the *sharp constant* or *Jackson constant* in Jackson inequality (1.1).

We recall only some fundamental results in Jackson inequalities concerning direct theorems. Firstly, we introduce some necessary notation. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_+$ be the set of all positive real numbers. Denote by $C(\mathbb{T})(\mathbb{T}=[-\pi,\pi])$ the space of continuous, 2π -periodic functions $f:\mathbb{R}\to\mathbb{R}$, with the uniform norm $\|f\|_{C(\mathbb{T})}=\max\{|f(x)|: x\in\mathbb{R}\}$, by $L^2(\mathbb{T})$ the space of real-valued, 2π-periodic, measurable functions which are square summable on $\mathbb{T}$ with the following $L^2(\mathbb{T})$—norm,

$$\|f\|_{L^2(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^2\, dx\right)^{1/2}, \tag{1.2}$$

by $L^2(\mathbb{R})$ the space of real-valued, measurable, square summable functions in the real line $\mathbb{R}$ with the $L^2(\mathbb{R})$-norm,

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2},$$

by $\mathscr{T}_n$ the set of all trigonometric polynomials of degree not higher than n, and by W_σ^2, $\sigma\geq 0$, the collection of all entire functions of exponential type σ which as functions of a real $x\in\mathbb{R}$ lie in $L^2(\mathbb{R})$.

Denote by X a normed space of some functions defined on $\mathbb{R}$ with the norm $\|\cdot\|_X$. For any $r\in\mathbb{N}$, the structure characterization of the function $f\in X$ is the modulus of continuity of order r of f:

$$\omega_r(f,\delta)_X = \sup\{\|\Delta_t^r f\|_X : t\in\mathbb{R}, |t|\leq\delta\}, \quad \delta\geq 0, \tag{1.3}$$

where

$$\Delta_t^r f(x) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(x+jt), \tag{1.4}$$

while $\binom{r}{0}=1$, $\binom{r}{j}=r(r-1)\cdots(r-j+1)/j!$, $j=1,2,\cdots,r$.

In the most cases, the mathematicians consider Jackson inequality (1.1) for the cases $X=C(\mathbb{T})$, $X=L^2(\mathbb{T})$ or $X=L^2(\mathbb{R})$, and correspondingly $L=\mathscr{T}_n\,(n\in\mathbb{N})$ or $L=W_\sigma^2\,(\sigma\in\mathbb{R}_+)$. In 1911, Jackson [15] proved the inequality (1.1) for the case $X=C(\mathbb{T})$, $L=\mathscr{T}_n$. He obtained that for any function $f\in C(\mathbb{T})$, the quantity $d(f,L,X)=E_n(f)_{C(\mathbb{T})}$ of the best uniform approximation of $f\in C(\mathbb{T})$ by trigonometric polynomials of order (at most) n tends to zero (as $n\to\infty$) not slower than $\omega_1(f,1/n)_{C(\mathbb{T})}$, which is defined as (1.3) and (1.4) with taking $X=C(\mathbb{T})$ and $r=1$. More precisely, the inequality

$$E_n(f)_{C(\mathbb{T})} \leq M_1\omega_1(f,1/n)_{C(\mathbb{T})}, \quad f\in C(\mathbb{T}), \quad n\geq 1,$$