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A Paradoxical Consistency Between Dynamic and Conventional Derivatives on Hybrid Grids

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Abstract. It has been evident that the theory and methods of dynamic derivatives are playing an increasingly important rôle in hybrid modeling and computations. Being constructed on various kinds of hybrid grids, that is, time scales, dynamic derivatives offer superior accuracy and flexibility in approximating mathematically important natural processes with hard-to-predict singularities, such as the epidemic growth with unpredictable jump sizes and option market changes with high uncertainties, as compared with conventional derivatives. In this article, we shall review the novel new concepts, explore delicate relations between the most frequently used second-order dynamic derivatives and conventional derivatives. We shall investigate necessary conditions for guaranteeing the consistency between the two derivatives. We will show that such a consistency may never exist in general. This implies that the dynamic derivatives provide entirely different new tools for sensitive modeling and approximations on hybrid grids. Rigorous error analysis will be given via asymptotic expansions for further modeling and computational applications. Numerical experiments will also be given.

AMS subject classifications: 34A45, 39A13, 74H15, 74S20

Key words: Dynamic derivatives, conventional derivatives, time scale theory, approximations, error estimates, hybrid grids, uniform and nonuniform grids, asymptotic expansions.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

There has been a considerable amount of recent research activities in the study of different types of dynamic equations as well as their computational applications via hybrid grids [2,3,5,6,13]. The most important issues in the theory and methods include unifying existing continuous and discrete representation methodologies, bridging the discrepancies between traditional differential and difference equations, and promoting highly efficient hybrid tools for mathematical modeling and scientific computations. Many interesting results have been obtained in this rapidly developing field [1,6,8–10,13]. Latest research in

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the subject has been extended to partial and high-order dynamic equations with sophisticated applications [2,6,8–10].

A dynamic derivative is a special rate of change formula defined on a hybrid grid. Different dynamic derivatives are used as building blocks for dynamic equations on hybrid grids. Therefore, it is extremely important to understand precise connections between different dynamic derivatives and conventional derivatives so that correct mathematical formulations can be constructed for modeling and approximation purposes [5,8,9].

It is not always easy, however, to investigate such a sensitive problem since different concerns and criteria may apply in different dynamic derivatives in the literature. In this discussion, we shall restrict ourselves to the issue of the consistency between the derivatives via standard numerical analysis. Without loss of generality, we shall only consider hybrid grids which are sets of real numbers superimposed upon nonempty bounded intervals. This is a natural extension of the pioneer exploration in [7]. The hybrid grids defined in this way can be viewed as generalizations of many popular irregular grids in applications, such as the moving and adaptive grids in quenching and blow-up solution computations [4,9–12].

Logically, we may expect a dynamic derivative defined on a hybrid set to be consistent with its conventional derivative counterpart on the interval, since both of the them are rate of change functions measuring variations of the targeted functions over the domains. We shall focus on the second-order Δ and ∇ dynamic derivatives and their crossed derivatives in this investigation [1–3,6,9]. Paradoxical relationships between the two derivatives over the hybrid grids will be discussed. We shall prove that the second-order dynamic derivatives are not consistent with conventional derivatives in general. However, interesting connections do exist between the two sides. Modifications of some dynamic derivative formulae may lead to good consistency. Proper incorporations of the underlying hybrid grid structures are often the keys to this success.

We would assume that the readers have a minimal working experience with the time scales theory and methods. Our approaches will be organized as follows. In Section 2, a brief introduction and a review of the dynamic derivatives as well as dynamic equations will be given. Concepts of approximations will also be established. Section 3 will be devoted to the study of the crossed second-order Δ, ∇ dynamic derivatives. We will then continue the exploration to non-crossed second-order dynamic derivatives in Section 4. The most paradoxical relations and numerical features from approximation point-of-view will be studied in these sections. Asymptotic expansions will be employed for deriving the local error estimates. A number of modified dynamic derivative formulae will be proposed on discrete hybrid grids. Finally, together with a number of straightforward numerical illustrations, our final conclusions and remarks will be delivered in Section 5.

2. Dynamic derivatives on hybrid grids

An *one-dimensional hybrid grid* $\mathscr{T}$ is defined as a nonempty closed subset of $\mathbb{R}$. Since $\mathscr{T}$ is bounded, we may set $a = \sup\mathscr{T}$, $b = \inf\mathscr{T}$ for the sake of convenience. In this case, a hybrid grid can be viewed as a closed set of real numbers superimposed over the