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Some Remarks on the Convex Feasibility Problem and Best Approximation Problem

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Abstract. In this paper we investigate several solution algorithms for the convex feasibility problem (CFP) and the best approximation problem (BAP) respectively. The algorithms analyzed are already known before, but by adequately reformulating the CFP or the BAP we naturally deduce the general projection method for the CFP from well-known steepest decent method for unconstrained optimization and we also give a natural strategy of updating weight parameters. In the linear case we show the connection of the two projection algorithms for the CFP and the BAP respectively. In addition, we establish the convergence of a method for the BAP under milder assumptions in the linear case. We also show by examples a Bauschke's conjecture is only partially correct.

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1. Introduction

The convex feasibility problem (CFP) is to find a point in the nonempty intersection $C=\bigcap_{i=1}^m C_i$ of a family of closed convex subsets $C_i \subseteq R^n, 1\le i\le m$, of the n-dimensional Euclidean space. It is a fundamental problem in many areas of mathematics and the physical sciences. More precisely, it has been used to model significant real-world problems including image reconstruction from projections, radiation therapy treatment planning, and crystallography (see [7] and the references therein). The convex sets $\{C_i\}_{i=1}^m$ represent mathematical constraints obtained from the modelling of the real-world problem.

The best approximation problem (BAP) is to find the projection of a given point $y\in R^n$ onto the nonempty intersection $C := \bigcap_{i=1}^m C_i \neq \emptyset$ of a family of closed convex subsets $C_i \subseteq R^n, 1\le i\le m$, i.e., we need to look for a point in C which is closest to y. The relevant

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background knowledge may consult [1] and [16]. In the CFP, any point in the intersection is acceptable to the real-world, while for the BAP it is appropriate if some point $y \in R^n$ has been obtained from modelling and computational efforts that initially did not take into account the constraints represented by the sets $\{C_i\}_{i=1}^m$ and now one wishes to incorporate them by seeking a point in the intersection of the convex sets which is closest to the point y.

For the CFP a number of solution methods have been presented (see [3, 8–11, 13, 14, 21–23, 27, 28, 30]). Among them, some are particularly designed for the CFP of special forms. Roughly speaking, these algorithms can be divided into two categories: projection method and interior method. For the BAP, several projection-type algorithms have been proposed to solve it. (see [2, 7, 16–18]).

The orthogonal projection $P_\Omega(z)$ of a point $z \in R^n$ onto a closed convex set $\Omega \subseteq R^n$ is a point of Ω defined by

$$P_\Omega(z) := arg\min\{\|z - x\|_2\},$$

where $\|.\|_2$ is the Euclidean norm in R^n.

The projection-type methods employ projection onto the individual convex sets in order to reach the required point in the intersection. Obviously the solution of the BAP for any given y is a solution of the CFP provided that $\bigcap_{i=1}^m C_i \neq \emptyset$. So it is easy to see that the iterate projection algorithms for the BAP are usually more complicated than algorithms for the CFP. However, we will show in Section 3 that at least in the linear case a relaxed projection algorithm for the CFP will produce a solution of the BAP as long as taking y as the starting point of the iteration.

In the present paper we intend to supply a relatively unified treatment for various projection algorithms for the CFP based on the steepest descent method. We also study the iterate behaviors of the sequential and simultaneous versions of Halpern-Lions-Wittmann-Bauschke (HLWB) algorithm for the BAP. In particular, we establish the convergence of the simultaneous HLWB in the linear case, which means that the algorithms can be accelerated. Moreover we show that when $\bigcap_{i=1}^m C_i = \emptyset$ a Bauschke's conjecture is only partially correct.

This paper is organized as follows. In Section 2, based on the reformulations of the CFP we naturally deduce the exact and surrogate relaxed projection algorithms for the CFP, from which we suggest a more natural updating strategy of weight parameters. In Section 3, we prove the convergence of the simultaneous HLWB algorithm in the linear case under mild conditions. In Section 4, we discuss the simultaneous HLWB in the case of intersection sets being empty and show that a conjecture due to Bauschke is only partially correct.

2. Several algorithms for the CFP

In this section three well-known algorithms for the CFP are further discussed.