Numer. Math. Theor. Meth. Appl. Vol. **12**, No. 2, pp. 348-369
doi: 10.4208/nmtma.OA-2018-0017 May 2019

An Adaptive Complex Collocation Method for Solving Linear Elliptic PDEs in Regular Convex Polygons Based on the Unified Transform

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Received 29 January 2018; Accepted (in revised version) 17 April 2018

Abstract. In this paper we present a novel approach for solving linear elliptic PDEs in regular convex polygons. The proposed algorithm relies on the so-called unified transform, or Fokas method. The basic step of this method involves the formulation of an equation coupling the finite Fourier transforms of the given boundary data and of the unknown boundary values, which is called the global relation. Herewith, a numerical scheme is proposed which computes the solution in the interior of a regular convex polygon using only the associated global relation. In particular, an adaptive complex collocation method is presented in order to solve numerically the global relation, using discrete boundary data. Additionally, the solution of a given PDE is computed in the entire computational domain, using a spatial-stepping scheme in conjunction with an adaptive complex collocation method. Moreover, a polynomial interpolation scheme is used near the center of the domain, and this increases the accuracy of the proposed method. We provide numerical results illustrating the applicability of the method as well as a comparison to a finite element formulation.

AMS subject classifications: 65N22, 65N35, 65R10

Key words: Elliptic PDEs, unified transform method, complex collocation method, spatial-stepping.

1. Introduction

Boundary value problems (BVPs) for linear partial differential equations (PDEs) can be solved via an integral transform method that was introduced in [7–10]. This method,

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which is known as the unified transform, or Fokas method, requires two basic steps. First, the solution is expressed in terms of integrals formulated in the complex Fourier space that involve certain Fourier-type integral transforms (containing complex valued parameters) of the Dirichlet and Neumann boundary values integrated along the boundary of the domain. These integral transforms are coupled via an algebraic equation known as the global relation, which defines a generalized DtN map. Then, the boundary values that are unknown can be determined using the fact that the global relation holds for all complex values λ. The above method can be considered as the spectral analogue of the classical boundary integral method, which is formulated in the physical instead of the spectral space.

For BVPs formulated in complicated domains and general boundary conditions the global relation cannot be solved in closed form. However, for certain BVPs for which classical methods have failed, analytical solutions based on the unified transform method have been derived [12]. For polygonal domains, the global relation has been solved numerically with a collocation-type technique introduced in [14] for a variety of BVPs [6,13,15,16,18]. Formulations based on Legendre polynomials can achieve spectral accuracy [13, 16]. The finite Fourier transform of Legendre polynomials can be computed by using the modified Bessel function [16] (which is computationally more efficient than using the closed form solution [11]). By oversampling the global relation in the complex λ-plane [13], and furthermore, by choosing an appropriate set of collocation points along certain rays [16], an overdetermined linear system can be derived with a low condition number. Solving this linear system yields the unknown boundary values.

In this paper, a novel numerical method is proposed for solving linear elliptic PDEs in the interior of convex regular polygons with an arbitrary number of sides. The key ingredient of this approach is an adaptive complex collocation scheme designed to be used with analytical and/or discrete boundary data. The solution is computed in a marching approach beginning from the boundaries and advancing towards the domain's central grid point, using a spatial-stepping scheme, based on Heun's method. By adaptively choosing the collocation points in the Fourier space at each spatial level we avoid the derivation of rank-deficient linear systems, keeping the marching procedure stable. Additionally, by using a polynomial interpolation scheme near the center of the domain, the accuracy of the approximated solution can be further improved.

It should be noted that an alternative approach based on the direct numerical evaluation of the integral representation has been demonstrated in [6]; however, according to the authors of [6], this implementation needs further development in order to become computationally practical. The combination of the analysis of the global relation and of the use of the classical Green's representation is advocated in [4].

There are several advantages associated with the unified transform over existing methods. The main advantage is that it is a boundary based method, thus the computational work is reduced. This is an important feature for problems with small surface-to-volume ratio. Contrary to the boundary element method, the unified transform does not rely on the availability of fundamental solutions, and furthermore, it does not require the computation of integrals that contain singularities. It is worth mentioning that a boundary integral formulation based on Legendre polynomials has been considered in [13], and it