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A New Variational Approach for Inverse Source Problems

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Abstract. We propose a new variational approach for recovering a general source profile in an elliptic system, using measurement data from the interior of the physical domain. The solution of the ill-posed inverse source problem is achieved by solving only one well-posed direct elliptic problem, resulting in the same computational cost as the one for the direct problem, and hence making the whole solution process of the inverse problem much less expensive than most existing methods. The resulting approximate solution is shown to be stable with respect to the change of the noise in the observation data, and a desired error estimate is also established in terms of the mesh size and the noise level in observation data. Numerical experiments are presented to confirm the theoretical predictions.

AMS subject classifications: 65N30, 65N55

Key words: Variational approach, inverse source problem, stability, convergence.

1. Introduction

In this work we are interested in a mathematical and numerical investigation of an elliptic inverse source problem [1, 8, 9, 19, 20]. Inverse source problems may appear in various mathematical PDE models, for example, parabolic equations [7,12,22], hyperbolic equations [18, 21], and Helmholtz equations [6, 10]. We refer to the monograph [19] for more introductions to inverse source problems. Pollutant source inversion problems have wide applications in the detection and monitoring of underground water pollution, indoor

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and outdoor air pollution, etc. Physical, chemical and biological measures have been developed for the identification of sources and source strengths [2, 36, 37]. We shall focus on the physical measures, that is, collecting pollutant concentration data measured by distributed sensors, then numerically reconstructing the source intensities at a priori known source locations with noise-contaminated data using mathematical models. The inverse source problem is ill-posed in the sense of Hadamard [11, 28, 33], especially the lack of stability with respect to the error in the measurement data is the most difficult challenge for numerical inversions, namely, sm-all change in the data may lead to significant difference in the reconstructed source strength. Inverse source problems have attracted a great attention, and numerous methods have been developed, including both deterministic and statistical methods [24, 31]. Quasi-explicit reconstruction formulas were worked out for one-dimensional source location recovery problems [15, 16], and quasi-reversibility methods were proposed to retrace the pollutant history [30]. Optimization based methods have been widely studied [4, 22, 23, 29, 32, 34]. By reformulating an inverse problem into an output least-squares PDE-constrained optimization problem complemented with Tikhonov regularization, classical optimization methods such as regression methods [14], linear and nonlinear programming methods [14], linear and nonlinear conjugate gradient methods [3, 32], Newton type methods, etc. can be applied for the approximate solutions of the optimisation systems.

In this work, we propose a new variational approach for solving an elliptic inverse source problem using measurement data from the interior of the physical domain. The inverse source problem is ill-posed mathematically [8, 19].

We shall construct a numerical method that needs only to solve one well-posed direct elliptic problem, so the new method has basically the same computational cost as the one for the direct problem, making the whole solution process of the inverse problem much less expensive than most existing methods, e.g., those aforementioned methods involving minimizations, particularly the standard Tikhonov regularization method (see (2.5)). The majority of the existing methods are of iterative type or involve the solution of a nonlinear minimization, for which at least two related PDEs need to be solved at each iteration, including a forward governing PDE and its associated adjoint PDE. The resulting approximate solution is shown to be stable with respect to the change of the noise in the observation data. More importantly, a desired error estimate is established in terms of the mesh size and the noise level in observation data. This appears to be another significant advantage of the new method, since no error estimates of the numerical solutions can be achieved in terms of the mesh size and noise level in the data for most existing methods.

The rest of this paper is arranged as follows. The next section discusses the concerned inverse source problem and its standard regularization method. A new variational method is proposed in Section 3, and its stability is established in Section 4. The finite element method and its convergence are investigated in Section 5. In Section 6, we derive an efficient numerical technique to help evaluate the negative norm of any finite element function. Numerical results are presented in Section 7 to verify the effectiveness of the new method.