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Weighted Integral of Infinitely Differentiable Multivariate Functions is Exponentially Convergent

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Abstract. We study the problem of a weighted integral of infinitely differentiable multivariate functions defined on the unit cube with the L_∞-norm of partial derivative of all orders bounded by 1. We consider the algorithms that use finitely many function values as information (called standard information). On the one hand, we obtained that the interpolatory quadratures based on the extended Chebyshev nodes of the second kind have almost the same quadrature weights. On the other hand, by using the Smolyak algorithm with the above interpolatory quadratures, we proved that the weighted integral problem is of exponential convergence in the worst case setting.

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1. Introduction

A multivariate numerical problem refers to a sequence of solution operators $S = \{S_d : F_d \to G_d\}_{d\in\mathbb{N}}$, where for each d, F_d is a class of functions with d variables and G_d is another space. Multivariate problems occur in many applications such as in computational finance, statistics and physics. To solve these solution operators, we often use information based algorithms that use finitely many information operations. Due to considering integral problem, in this paper, we only allow any function values to be an information operation.

Most of the work on multivariate computational problems has dealt with problems defined over classes of functions with finite smoothness. For such problem classes, the corresponding minimal error sequence often converges polynomially. However, there has been recent work in the worst case setting (see, e.g., [1-4]) on problems having infinite smoothness, including problems defined over spaces of analytic functions. For such problem classes, the convergence rate of the minimal error sequence will often be faster than

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polynomial (e.g., supper-polynomial or exponential). Noted that all the problems involved in above papers are over the weighted reproducing kernel Hilbert spaces, K. Suzuki [5] focused on the integral problem on a weighted L_1-normed space which consists of non-periodic smooth functions. It used the multivariate QMC rules on digital nets to prove that the corresponding minimal error sequence converges supper-polynomially. In this paper, we will consider a weighted integral problem on the following infinitely differentiable function class that was introduced in [6]:

$$F_d = \Big\{ f : [-1,1]^d \to \mathbb{R} \Big| \|f\|_{F_d} = \sup_{\alpha=(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}_0^d} \|D^\alpha f\|_\infty < \infty \Big\}, \tag{1.1}$$

and we proved that the corresponding minimal error sequence converges exponentially. The reason that we use $[-1,1]$ instead of $[0,1]$ in [5, 6] is that we will use the Chebyshev nodes. We mainly used the Smolyak algorithm and the interpolatory quadratures based on the extended Chebyshev nodes. We would like to add that our proof is also suitable for non-weighted integral.

The paper is organized as follows. Section 2 contains some basic concepts and lemmas that will be needed in the proofs of our main results. In Section 3 we give the main results and their proofs.

2. Some concepts and lemmas

First, let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ respectively denote the sets of all positive integers, non-negative integers and real numbers.

Now we introduce the related concepts. Assume that each operator $S_d\colon F_d \to G_d$ is a continuous linear transformation, where F_d is a Banach space of d-variate real functions defined on $D_d \subset \mathbb{R}^d$ and G_d is another Banach space.

For each $d \in \mathbb{N}$, we consider the approximation of $S_d(f)$ for $f \in F_d$ by using *information-based algorithms* of the form

$$A_{n,d}(f) = \phi_{n,d}\left(L_1(f), \cdots, L_n(f)\right), \tag{2.1}$$

where $L_1, L_2, \cdots, L_n \in \Lambda^{\mathrm{std}} = \{L | L(f) = f(t),\ \forall t \in D_d\}$ and $\phi_{n,d} : \mathbb{R}^n \to G_d$ is an arbitrary mapping. As a special case, we define $A_{0,d} = 0$.

The worst-case error of the algorithm $A_{n,d}$ is defined as

$$e(S_d, A_{n,d}, F_d, G_d) = \sup_{f\in F_d, \|f\|_{F_d}\le 1} \left\| S_d(f) - A_{n,d}(f) \right\|_{G_d}.$$

Furthermore, we define the nth minimal worst-case error as

$$e(n, S_d, F_d, G_d) = \inf_{A_{n,d}} e(S_d, A_{n,d}, F_d, G_d),$$

where the infimum is taken over all algorithms of the form (2.1).