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A Consistent Fourth-Order Compact Finite Difference Scheme for Solving Vorticity-Stream Function Form of Incompressible Navier-Stokes Equations

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Abstract. The inconsistent accuracy and truncation error in the treatment of boundary usually leads to performance defects, such as decreased accuracy and even numerical instability, of the entire computational method, especially for higher order methods. In this work, we construct a consistent fourth-order compact finite difference scheme for solving two-dimensional incompressible Navier-Stokes (N-S) equations. In the proposed method, the main truncation error term of the boundary scheme is kept the same as that of the interior compact finite difference scheme. With such a feature, the numerical stability and accuracy of the entire computation can be maintained the same as the interior compact finite difference scheme. Numerical examples show the effectiveness and accuracy of the present consistent compact high order scheme in L^∞ . Its application to two dimensional lid-driven cavity flow problem further exhibits that under the same condition,the computed solution with the present scheme is much close to the benchmark in comparison to those from the 4^{th} order explicit scheme. The compact finite difference method equipped with the present consistent boundary technique improves much the stability of the whole computation and shows its potential application to incompressible flow of high Reynolds number.

AMS subject classifications: 65Y04, 65Z05, 65N06, 65D25
Key words: Navier-Stokes equations, compact finite difference scheme, consistent boundary scheme, Lid-driven cavity.

1. Introduction

It is desirable to use higher order numerical methods to solve complex flow problems in many applications, such as those governed by incompressible Navier-Stokes (N-S) equations, due to their lower numerical diffusion and dispersion. Because of its compact

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stencils among high order methods, there is consistent interest in developing compact finite difference method for solving convection-diffusion problems and N-S equations [1–7] since Adam, Kreiss and Hirsh [8] put forward the Hermitian three-point fourth-order compact scheme. One of the followed pioneering work was done by Lele [9], where a class of high order centered and linear compact schemes was proposed with high order boundary schemes suggested. A centered compact scheme is usually very effective for solving diffusion dominant problems with uniform mesh but encounters difficulty when applied to convection dominant problems.

With this regard, Fu and Ma [10] investigated the upwind compact finite difference approximation for solving convective dominant equations. Later, Tian [13] explored a kind of fourth-order explicit upwind compact difference schemes. In order to apply a compact scheme to nonuniform mesh, Chu and Fan [11,12] tried to construct a sixth-order compact scheme via jointing the uniform and non-uniform grids. And Zhang et al. [14] developed a sixth-order compact scheme on staggered grids.

High-order compact finite difference schemes require additional numerical boundary schemes to treat the grid points near boundaries of the computational domain. Although it has proven that for a pth-order scheme,the accuracy of boundary scheme can be one order lower, i.e. $(p-1)$th-order, than that of the interior scheme in order to maintain the global accuracy of the entire computation [16] under L^2-norm, boundary treatment is still the major challenging of applying high order methods in engineering applications. The primary difficulty in using higher order compact finite difference schemes is to identify boundary schemes that are able to preserve the accuracy and the stability as well over the whole computational domain. For hyperbolic systems, Carpenter [17] introduced the fourth and sixth order compact schemes with Lele's boundary scheme [9] and a proposed sixth order boundary scheme, respectively. Recently, Liu et al. [15] also exhibited a class of centered compact finite difference schemes and the corresponding boundary schemes. For the two-dimensional vorticity-stream function form of incompressible N-S equations, E and Liu [18, 19] put forward a kind of fourth-order accuracy schemes, with both interior and boundary schemes to fourth-order accuracy.

We noticed that in all of those work mentioned above, the authors focused on developing some boundary schemes with the same order accuracy as that of the interior scheme, they did not pay much attention on numerical stability of the entire method. We have found recently that some of the popular boundary schemes can suffer numerical instability when applied to flow in high Reynolds number and the numerical stability of the entire computation is closely related to the coefficients of the leading truncation errors of both interior and boundary schemes.

This paper is primarily aimed at developing a kind of consistent fourth-order compact finite difference scheme to solve vorticity-stream function form of the two-dimensional incompressible Navier-Stokes equations. We present a fourth-order compact scheme for the boundary computation, in which the order of accuracy and leading truncation error term is designed to be the same as that of the interior scheme so that both the accuracy and stability can be kept during the whole computation.